Math 245C Lecture Notes

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Contents

1 Functions of Bounded Variation and Distribution Functions

1.1 Functions of bounded variation

First, let's review the idea of functions of bounded variation.

Definition 1.1. Let $-\infty < a < b < \infty$. We say that $f : [a,b] \to \mathbb{R}$ is of **bounded variation** and write $f \in \mathrm{BV}([a,b])$ if

$$\sup_n \sup_{x_i} \left\{ \sum_{i=1}^{n-1} |f(x_i) - f(x_{i-1})| : a = x_0 < x_1 < \cdots < x_n = b \right\} < \infty$$

We call this supremum the **total variation norm** and write it as $\|f\|_{\mathrm{TV}([a,b])}$.

If $f : [a,b] \to \mathbb{R}$, we write $f' = f'_{\text{abs}} + f'_{\text{sing}}$, where $\int |f'_{\text{abs}}| + \int |f'_{\text{sing}}| < \infty$.

Definition 1.2. We sat that $F : \mathbb{R} \to \mathbb{C}$ is of bounded variation if

$$\sup_{x_0, x_1} \left\{ \|F\|_{\mathrm{TV}([x_0,x_1])} : -\infty < x_0 < x_1 < \infty \right\} < \infty.$$

Set $T_F(x) = \sup_{x_0 < x} \|F\|_{\mathrm{TV}([x_0,x])}$. This is a monotone increasing function. Observe that $F \in \mathrm{BV}(\mathbb{R})$ means that $\lim_{x \to \infty} T_F(x) < \infty$.

We can normalize functions of bounded variation.

Definition 1.3. $\mathrm{NBV}(\mathbb{R})$ is the set of $F \in \mathrm{BV}(\mathbb{R})$ such that

1. F is right continuous.
2. $\lim_{x \to -\infty} F(x) = 0$.

Definition 1.4. If ν_1, ν_2 are two signed Borel measures on $\mathbb{R}$ of finite total mass, $\nu = \nu_1 + i\nu_2$ is called a **complex Borel measure**.

Remark 1.1. Signed measures can take the values $\pm\infty$, but we require them to be finite here.

Proposition 1.1. *If $F \in \mathrm{NBV}(\mathbb{R})$, then there exists a unique Borel complex measure μ_F on $\mathbb{R}$ such that $F(x) = \mu_F((-\infty, x])$. Conversely, every Borel complex measure is of the form μ_F.*

Theorem 1.1 (integration by parts)**.** *Let $F, G \in \mathrm{BV}([a,b])$, where $-\infty < a < b < \infty$. Assume F is right continuous and G is continuous. Then*

$$\int_{(a,b]} F(x)\, d\mu_G(x) + \int_{(a,b]} G(x)\, d\mu_F(x) = F(b)G(b) - F(a)G(a).$$

Remark 1.2. One uses the notation

$$\int_{(a,b]} F(x) \mu_G(x) = \int_{(a,b]} F(x)\, dG(x).$$

1.2 Distribution functions

Throughout this section, $(X, \mathcal{M}, \mu)$ is a measure space, and $0 < p < \infty$.

Definition 1.5.

$$L^p(X, \mu) = \left\{ F : X \to \mathbb{C} : F \text{ is measurable}, \int_X |F|^p \, d\mu < \infty \right\}.$$

We write

$$\|F\|_{L^p} = \left(\int_X |F(x)|^p \, d\mu(x) \right)^{1/p}.$$

Remark 1.3. We will write L^p or $L^p(\mu)$ for $L^p(X, \mu)$.

Proposition 1.2 (Chebyshev's inequality)**.** *Fix $\alpha > 0$.*

$$\int_X |F(x)|^p \, d\mu(x) \geq \alpha^p \mu(\{|F| > \alpha\}).$$

Proof.

$$\int_X |F(x)|^p \, d\mu(x) \geq \int_{\{|F|>\alpha\}} |F(x)|^p \, d\mu(x) \geq \int_{\{|F|\geq\alpha\}} \alpha^p \, d\mu(x) \geq \alpha^p \mu(\{|F| > \alpha\}). \quad \square$$

Remark 1.4. If $F \in L^p$, then

$$\sup_{\alpha>0} \alpha^p \mu(\{|F| > \alpha\}) \leq \|F\|_{L^p}^p < \infty.$$

Definition 1.6. Let $F : X \to \mathbb{C}$ be measurable. The **distribution function** of F is $\lambda_F : (0, \infty) \to [0, \infty]$ defined as $\lambda_F(\alpha) = \mu(\{|F| > \alpha\})$.

Proposition 1.3. *Let $F, G :\to \mathbb{C}$ be measurable.*

1. *λ_F is monotone decreasing.*
2. *If $|F| \leq |G|$, then $\lambda_F \leq \lambda_G$.*
3. *If $H := F + G$, then $\lambda_H(\alpha) \leq \lambda_F(\alpha/2) + \lambda_G(\alpha/2)$.*
4. *If $F_n : X \to \mathbb{C}$ are measurable functions such that $|F_n| \leq |F_{n+1}| \leq |F|$ for all n, and $\lim_n |F_n| = |F|$, then $\lim_n \lambda_{F_n} = \lambda_F$.*

Proof. Define $E(\alpha, F) = \{|F| > \alpha\}$ for $\alpha > 0$.

1. If $0 < \alpha_1 < \alpha_2$, then $E(\alpha_2, F) \subseteq E(\alpha_1, F)$. So

$$\lambda_F(\alpha_2) = \mu(E(\alpha_2, F)) \leq \mu(E(\alpha_1, F)) = \lambda_F(\alpha_1).$$

2. If $|F| \leq |G|$, then for $\alpha > 0$, $E(\alpha, F) \subseteq E(\alpha, G)$.

3. If $|H| > \alpha$, then $|F| + |G| \geq |F + G| = |H| > \alpha$. Then $|F| > \alpha/2$ or $|G| > \alpha/2$. So $E(\alpha, H) \subseteq E(\alpha/2, F) \cup E(\alpha/2, G)$. So

$$\mu(E(\alpha, H)) \leq \mu(E(\alpha/2, F)) + \mu(E(\alpha/2, G)).$$

4. Let $(F_n)_n$ be as above. Then $\lambda_{F_n} \leq \lambda_{F_{n+1}} \leq \lambda_F$. Hence, $\lim_n \lambda_{F_n}$ exists and is $\leq \lambda_F$. To get the reverse inequality, we use

$$E(\alpha, F) = \bigcup_{n=1}^{\infty} E(\alpha, F_n).$$

To get the $\subseteq$ containment, if $|F(x)| > \alpha$, then there exists n such that $|F_n(x)| > \alpha$. Note that $E(\alpha, F_n) \subseteq E(\alpha, F_{n+1}) \subseteq E(\alpha, F)$ for all n. Since μ is a measure,

$$\mu(E(\alpha, F)) = \mu\left(\bigcup_{n=1}^{\infty} E(\alpha, F_n)\right) = \lim_n \mu(E(\alpha, F_n)). \qquad \square$$

Definition 1.7. **Weak** L^p, denoted $L^p(\mu, \text{weak})$, os the set of measurable functions $F : X \to \mathbb{C}$ such that $[F]_p < \infty$, where

$$[F]_p = \sup_{\alpha \in (0, \infty)} \alpha^p \lambda_F(\alpha).$$

Remark 1.5. $L^p(\mu) \subseteq L^p(\mu, \text{weak})$.

These are not the same. What is the difference? We will show that being in weak L^p is equivalent to $\int_0^\infty \alpha^{p-1} \lambda_F(\alpha)\, d\alpha < \infty$. So $F \in L^p$ means that $\alpha^{p-1}\lambda_F \in L^1((0, \infty))$, while $F \in L^p(\mu, \text{weak})$ means that $\alpha^p \lambda_F \in L^\infty(0, \infty)$.

2 Integration With Push-Forward Measures and Distribution Functions

2.1 Integration with push-forward measures

Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $0 < p < \infty$.

Definition 2.1. Let $(Y, \mathcal{N}, \nu)$ be another measure space, and let T be a measurable map. We say that T **pushes** μ **forward to** ν if $\nu(B) = \mu(T^{-1}(B))$ for all $B \in \mathcal{N}$.

Proposition 2.1. *T pushes μ forward to ν if and only if*

$$\int_Y f \, d\nu = \int_X f \circ T \, d\mu,$$

for all $f \in L^1(\nu)$.

Proof. We can restate the condition in the definition as

$$\int_Y f \, d\nu = \int_X f \circ T \, d\mu,$$

where $f = \mathbb{1}_B$. By linearity, this holds for when f is a simple function. This means that if $f : Y \to [0, \infty]$ is ν-measurable, then $\int_Y f \, d\nu = \int_X f \circ T \, d\mu$. By linearity, this holds for all $f \in L^1$. $\square$

Recall that if $F \in \mathrm{NBV}(\mathbb{R})$, there exists a unique Borel complex measure such that $\mu_F((-\infty, x]) = F(x)$.

Proposition 2.2. *Assume $f : X \to \mathbb{C}$ is measurable and $\lambda_f(\alpha) < \infty$ for all $\alpha > 0$. If $\phi : (0, \infty) \to \mathbb{R}$ is Borel, then*

$$\int_X \phi \circ |f| \, d\mu = \int_0^\infty \phi(\alpha) \, d\mu_{-\lambda_f}(\alpha).$$

In other words, $|f|_* \mu = \mu_{-\lambda_f}$.

Proof. It suffices to show the proposition when $\phi = \mathbb{1}_E$ and $E \subseteq (0, \infty)$ is Borel. In fact, it is not a loss of generality to further assume $E = (a, b]$, where $-\infty < a < b < \infty$. We need to check that $\mu_{-\lambda_f}(E) = \mu(x : \{|f(x)| \in E\})$. We have

$$\begin{aligned}
\mu(\{x : |f(x)| \in E\}) &= \mu(\{x : a < |f(x)| \leq b\}) \\
&= \mu(\{x : a < |f(x)|\}) - \mu(\{x : b < |f(x)|\}) \\
&= \lambda_f(a) - \lambda_f(b) \\
&= \mu_{-\lambda_f}((a, b]) \\
&= \mu_{-\lambda_f}(E).
\end{aligned}$$

$\square$

2.2 Integration with respect to distribution functions

Proposition 2.3. *Let $f : X \to \mathbb{C}$ be a simple function such that $f \in L^p(\mu)$.*

1. *For all $0 < \varepsilon_1 < \varepsilon_2$, $\lambda_f \in \mathrm{BV}([\varepsilon_1, \varepsilon_2])$.*

2. $$\int_X |f|^p \, d\mu(x) = p \int_0^\infty \alpha^{p-1} \lambda_f(\alpha)\, d\alpha.$$

Here is a wrong proof: Let $\phi(t) = |t|^p$. Then, using integration by parts,

$$\int_X \phi(|f|)\, d\mu = \int_0^\infty \phi(\alpha)\, d(-\lambda_f) = -\int_0^\infty \underbrace{\phi'(\alpha)}_{p\alpha^{p-1}}(-\lambda_f)\, d\alpha + -\lambda_f \phi|_0^\infty .$$

Proof. Write $f = \sum_{i=1}^n a_i \mathbb{1}_{A_i}$, where the A_i are measurable and pairwise disjoint. We can also assume a_i are distinct. We have $|f|^p = \sum_{i=1} |a_i|^p \mathbb{1}_{A_i}$, so

$$\sum_{i=1}^n |a_i|^p \mu(A_i) = \int_X |f|^p \, d\mu < \infty.$$

Let $I = \{a_i : a_i \neq 0\}$. Then $\|f\|_{L^p}^p \geq |a_i|^p \mu(A_i)$ for all $a_i \in I$. So

$$\mu\left(\bigcup_{a_i \in I} A_i\right) \leq \|f\|_{L^p}^p \sum_{a_I \in I} \frac{1}{|a_i|^p} =: \gamma.$$

If $\alpha > \max_{i=1,\dots,n} |a_i| := \overline{\gamma}$, then $\lambda_f(\alpha) = 0$. If $\alpha > 0$, $\{|f| < \alpha\} \subseteq \bigcup_{a_i \in I} A_i$, so $\lambda_f(\alpha) \leq \gamma$. If $\varepsilon_1 < \varepsilon_2 < \infty$, then $\lambda_f|_{[\varepsilon_1, \varepsilon_2]}$ has range contained in $[0, \gamma]$. This proves that $\lambda_f \in \mathrm{BV}([\varepsilon_1, \varepsilon_2])$.

Let $b < \overline{\gamma}$. Then by the previous proposition,

$$\begin{aligned}
\int_X |f|^p \, d\mu &= \int_0^\infty \alpha^p \, d\mu_{-\lambda_f}(\alpha) \\
&= \int_0^b \alpha^p \, d\mu_{-\lambda_f}(\alpha) \\
&= \lim_{\varepsilon_1 \to 0} \int_{\varepsilon_1}^b \alpha^p \, d(-\lambda_f) \\
&= \lim_{\varepsilon_1 \to 0} -\int_{\varepsilon_1}^b \alpha^{p-1}(-\lambda_f)(\alpha)\, d\alpha + \cancel{[-\alpha^p \lambda_f(\alpha)]_{\varepsilon_1}^b} \\
&= p \int_0^b \alpha^{p-1} \lambda_f(\alpha)\, d\alpha.
\end{aligned}$$

Indeed, since λ_f is bounded, $\lim_{\alpha \to 0} \alpha^p \lambda_f(\alpha) = 0$. $\square$

Corollary 2.1. *Let $f \in L^p(\mu)$. Then*

$$\int_X |f|^p \, d\mu = p \int_0^\infty \alpha^{p-1} \lambda_f(\alpha) \, d\alpha.$$

Proof. Let $f_n : X \to \mathbb{C}$ be a sequence of simple functions such that $|f_n| \leq |f_n| \leq |f|$ for all n and $|lim_n|f_n| = |f|$. By the previous proposition,

$$\int_X |f_n|^p \, d\mu = p \int_0^\infty \alpha^{p-1} \lambda_{f_n}(\alpha) \, d\alpha.$$

Since $\lambda_{f_n} \leq \lambda_{f_{n+1}} \leq \lambda_f$ and $\lim_n \lambda_{f_n} = \lambda_f$, we apply the dominated convergence theorem to conclude the proof, $\square$

3 Cutoff functions, The Riesz-Thorin Theorem, and Strong and Weak Type

3.1 Cutoff functions

Definition 3.1. For $f : \mathbb{C} \to \mathbb{R}$, $A > 0$, the **cutoff** function $\phi_A \in C(\mathbb{C}, \mathbb{C})$ is

$$\phi_A(z) = \begin{cases} z & |z| < A \\ Az/|z| & |z| > A. \end{cases}$$

Note that $\phi_A(\mathbb{C}) = \overline{B_A(0)}$ and $\phi_A(\mathbb{R}) \subseteq \mathbb{R}$.

Theorem 3.1. *Let $f : X \to \mathbb{C}$ be measurable, and for $A > 0$, set*

$$h_A = \phi_A \circ f, \qquad g_A = f - h_A.$$

Then

$$\lambda_{h_A}(\alpha) = \begin{cases} \lambda_f(\alpha) & \alpha < A \\ 0 & \alpha \geq A \end{cases}, \qquad \lambda_{g_A}(\alpha) = \lambda_f(\alpha + A).$$

Proof. Let $\alpha > 0$. Since $|h_A| \leq A$, $\{h_A > \alpha\} = \varnothing$ if $\alpha \geq A$. This shows $\lambda_{h_A}(\alpha) = 0$. If $0 < \alpha < A$, then $\{|h_A| > \alpha\} = \{|f| > \alpha\}$, so $\lambda_{h_A}(\alpha) = \lambda_f(\alpha)$.

Note that

$$g_A = f - \varphi_A \circ f \implies |g_A| = |f - \phi_A \circ f| = \begin{cases} 0 & |f| < A \\ |f - \frac{f}{|f|}A| & |f| > A. \end{cases}$$

Hence,

$$|g_A| = \begin{cases} 0 & |f| < A \\ |f| - A & |f| \geq A. \end{cases}$$

So if $\alpha > 0$, then $\{|g_A| > \alpha = \{|f| - A > \alpha\} = \{|f| < \alpha + A\}$. $\square$

3.2 The Riesz-Thorin interpolation theorem

Throughout this section (interpolation of L^p spaces), $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are measure spaces.

Let $p < q < r$. If $t \geq 0$, then

$$t^q \leq \begin{cases} t^r & t \geq 1 \\ t^p & 0 \leq 0 \leq 1. \end{cases}$$

So for any $t \in \mathbb{R}$, $|t|^q \leq |t|^p + |t|^r$ for all t. Hence, if $f : X \to \mathbb{C}$ is μ-measurable, then $|f|^q \leq |f|^r + |f|^p$. We get the following.

Proposition 3.1. $L^r(\mu) \cap L^p(\mu) \subseteq L^q(\mu)$.

Recall that ν is called **semifinite** if for any $E \in \mathcal{N}$ such that $\nu(E) = \infty$, there exists $F \in \mathcal{N}$ such that $F \subseteq E$ and $0 < \nu(F) < \infty$.

Theorem 3.2 (Riesz-Thorin interpolation theorem)**.** *Let $1 \leq p_0, q_0, p_1, q_1 < \infty$, and further assume that ν is semifinite if $q_0 = q_1 = \infty$. For $t \in (0,1)$, define p_t and q_t as*

$$\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1}, \qquad \frac{1}{q_t} = \frac{1-t}{q_0} + \frac{t}{q_1}.$$

Assume $T : L^{p_0}(\mu) + L^{p_1}(\mu) \to L^{q_0}(\nu) + L^{q_1}(\nu)$ is a linear operator such that there are M_0,$M_1 \geq 0$ such that

$$\|Tf\|_{L^{q_0}(\nu)} \leq M_0 \|f\|_{L^{p_0}(\mu)}, \qquad \|Tg\|_{L^{q_1}(\nu)} \leq M_1 \|g\|_{L^{p_1}(\nu)}$$

for every $f \in L^{p_0}(\mu)$ and $g \in L^{p_1}(\mu)$. Then

$$\|Th\|_{L^{q_t}(\nu)} \leq M_0^{1-t} M_1^t \|h\|_{L^{p_t}(\mu)}$$

for all $h \in L^{p_t}(\mu)$.

Remark 3.1. It it not surprising that this is bounded. The particular bound is the important part.

$$|Th|^{q_t} \leq |Th|^{q_0} + |Th|^{q_1},$$

so

$$\|Th\|_{L^{q_t}}^{q_t} \leq \|Th\|_{L^{q_0}}^{q_0} + \|Th\|_{L^{q_1}}^{q_1} \leq M_0^{q_0} \|h\|_{L^{p_0}}^{q_0} + M_1^{q_1} \|h\|_{L^{p_1}}^{q_1}.$$

We will not prove this theorem, as it involves a lemma that is technical and not very instructive.

3.3 Strong type and weak type

Let $\mathcal{D}$ be a vector subspace of the set of $(X, \mathcal{M}, \mu)$ measurable functions, and let $\mathcal{F}$ be the set of $(Y, \mathcal{N}, \nu)$ measurable functions.

Definition 3.2. We say that $T : \mathcal{D} \to F$ os sublinear if

1. $|T(f+g)| \leq |Tf| + |Tg|$
2. $|T(cf)| = c|Tf|$

for all $f, g \in \mathcal{D}$ and $c \geq 0$.

Definition 3.3. Let $T : \mathcal{D} \to \mathcal{F}$ be a sublinear map, and let $1 \leq p, q \leq \infty$. We say that T is (p, q)**-strong type** if there exists $c > 0$ such that

$$\|Tf\|_{L^q} \leq c\|f\|_{L^p}$$

for all $f \in \mathcal{D}$. We say that T is (p, q)**-weak type** if there exists $c > 0$ such that

$$[Tf]_q \leq c\|f\|_{L^p}$$

for all $f \in \mathcal{D}$, provided that $q < \infty$. We say that T is (p, ∞)**-weak type** if T is (p, ∞)-strong type.

Remark 3.2. If $f \in \mathcal{D}$ but $f \notin L^p(\mu)$, then the right hand side is ∞, satisfying the inequality. So we could replace the condition with $f \in L^p(\mu)$.

We can rewrite the strong type condition as

$$q \int_0^\infty \alpha^{q-1} \lambda_{T(f)}(\alpha)\, d\alpha \leq c^q \left(p \int_0^\infty \alpha^{p-1} \lambda_f(\alpha)\, d\alpha \right)^{q/p}.$$

Proposition 3.2. *Assume $f : X \to \mathbb{C}$ is measurable and $1 \leq p < \infty$. Then*

$$\|f\|_{L^p}^p = p \int_0^\infty \alpha^{p-1} \lambda_f(\alpha)\, \alpha.$$

Proof. If $f \in L^p(\mu)$, we have already proved this. Otherwise, suppose $f = \sum_{i=1}^n a_i \mathbb{1}_{A_i}$, where the A_i are measurable and pairwise disjoint. Since

$$\infty = \|f\|_{L^p}^p = \sum_{i=1}^n |a_i|^p \mu(A_i),$$

there is i such that $\mu(A_i) = \infty$. We have $\lambda_f \geq \lambda_{|a_i| \mathbb{1}_{A_i}}$. But $\lambda_{|a_i| \mathbb{1}_{A_i}}(\alpha) = \infty$ if $\alpha \in (0, |a_i|)$. Now for general f, approximated it form below by step functions and apply the dominated convergence theorem. $\square$

4 Minkowski's Inequality and The Marcinkiewicz Interpolation Theorem

4.1 Minkowski's inequality

Let $f : X \to \mathbb{C}$ be measurable. For $A > 0$, set $h_A = \phi_A \circ f$, $g_A = f - h_A$, where

$$\phi_A(z) = \begin{cases} z & |z| < A \\ \frac{z}{|z|}A & |z| \geq A. \end{cases}$$

Then

$$\lambda_{h_A}(\alpha) = \begin{cases} \lambda_f(\alpha) & 0 < \alpha < A \\ 0 & \alpha > A, \end{cases}, \qquad \lambda_{g_A}(\alpha) = \lambda_f(A + \alpha).$$

Recall Minkowski's inequality:

Theorem 4.1 (Minkowski's inequality)**.** *Let $1 \leq r < \infty$, and let $f : X \times Y \to [0, \infty]$. Then*

$$\int_Y \left(\int_X |f(x,y)|^r \, d\mu(x) \right)^{1/r} d\nu(y) \geq \left(\int_X \left(\int_Y f(x,y) \, d\nu(y) \right)^r d\mu(x) \right)^{1/r}.$$

4.2 The Marcinkiewicz interpolation theorem

Theorem 4.2 (Marcinkiewicz interpolation theorem)**.** *Let $\mathfrak{F}$ be the set of measurable functions on Y. Let $1 \leq p_0, p_1, q_0, q_1 \leq \infty$ be real numbers such that $p_0 \leq q_0$, $p_1 \leq q_1$, and $q_0 \neq q_1$. Let $t \in (0,1)$, and let p, q be defined as*

$$\frac{1}{p} = \frac{1-t}{p_0} + \frac{t}{p_1}, \qquad \frac{1}{q} = \frac{1-t}{q_0} + \frac{t}{q_1}.$$

Assume that $T : L^{p_0}(\mu) + L^{p_1}(\mu) \to \mathcal{F}$ be sublinear and of weak type (q_0, p_0) and (q_1, p_1) (there are $c_0, c_1 > 0$ such that if $q_0, q_1 \neq \infty$, $(\alpha^{q_0} \lambda_{T(f)})^{1/q_0} \leq c_0 \|f\|_{p_0}$ and $(\alpha^{q_1} \lambda_{T(f)})^{1/q_1} \leq c_1 \|f\|_{p_1}$). Then the following hold:

1. *T is strong type (p, q) (there exists $B_p > 0$ such that $\|Tf\|_q \leq B_p \|f\|_p$ for all $f \in L^p(\mu)$).*
2. *If $p_0 < \infty$, then $\lim_{p \to p_0} B_p |p_0 - p| < \infty$. If $p_1 < \infty$, then $\lim_{p \to p_1} B_p |p_1 - p| < \infty$. If $p_0 = \infty$, (B_p) remains bounded as $p \to p_0$. If $p_1 = \infty$, (B_p) remains bounded as $p \to p_1$.*

Proof. We skip the proof in the case $p_1 = p_0$. Let us assume $q_0, q_1 < \infty$.

Consider

$$\frac{p_0}{q_0}\frac{q-q_0}{p-p_0} = \frac{p_0}{q_0}\frac{q_0}{p_0}\frac{\frac{q}{q_0}-1}{\frac{p}{p_0}-1} = \frac{q}{p}\cdot\frac{\frac{1}{q_0}-\frac{1}{q}}{\frac{1}{p_0}-\frac{1}{p}} = \frac{q}{p}\cdot\frac{\frac{1}{q_0}-(\frac{1-t}{q_0}+\frac{t}{q_1})}{\frac{1}{p_0}-(\frac{1-t}{p_0}+\frac{t}{p_1})} = \frac{q}{p}\frac{\frac{1}{q_0}-\frac{1}{q_1}}{\frac{1}{p_0}-\frac{1}{p_1}}.$$

Also consider

$$\frac{p_1}{q_1}\frac{q-q_1}{p-p_1} = \frac{q}{p}\frac{\frac{1}{q_0}-\frac{1}{q_1}}{\frac{1}{p_0}-\frac{1}{p_1}}.$$

Set

$$r = \frac{p_0}{q_0}\frac{q-q_0}{p-p_0} = \frac{p_1}{q_1}\frac{q-q_1}{p-p_1}.$$

We have

$$\|g_A|_{L^{p_0}}^{p_0} = p_0\int_0^\infty \alpha^{p_0-1}\lambda_{g_A}(\alpha)\,d\alpha = p_0\int_0^\infty \alpha^{p_0-1}\lambda_f(A+\alpha)\,d\alpha = p_0\int_A^\infty (\beta-A)^{p_0-1}\lambda_f(\beta)\,d\beta$$

So

$$\|g_A\|_{p_0}^{p_0} \le p_0\int_A^\infty \beta^{p_0-1}\lambda_f(\beta)\,d\beta.$$

We have

$$\|h_A\|_{p_1}^{p_1} = p_1\int_0^\infty \lambda_{h_A}(\alpha)\alpha^{p_1-1}\,d\alpha = p_1\int_0^A \lambda_f(\alpha)\alpha^{p_1-1}\,d\alpha.$$

We also have

$$\|Tf\|_{q_0}^q = q\int_0^\infty \alpha^{q-1}\lambda_{T(f)}(\alpha)\,d\alpha = q\int_0^\infty (2\beta)^{q-1}\lambda_{Tf}(2\beta)\,d(2\beta).$$

Since $f = g_A + h_A$, we get that $|Tf| = |T(g_A+h_A)| \le |Tg_A| + |Th_A|$, So

$$\lambda_{|Tf|}(2\beta) \le \lambda_{Tg_A}(\beta) + \lambda_{Th_A}(\beta).$$

This lets us get

$$\|Tf\|_{q_0}^q \le 2^{q-1}q\int_0^\infty \beta^{q-1}\left(\lambda_{Tg_A}(\beta)+\lambda_{Th_A}(\beta)\right)\,d\beta.$$

Use the weak-type condition with f replaced by g_A and with f replaced by h_A to conclude that

$$\begin{aligned}\|Tf\|_{L^q}^q &\le 2^{q-1}q\int_0^\infty \alpha^{q-1}\left(\left(\frac{c_0}{\alpha}\right)_0^q\|g_A\|_{p_0}^{q_0} + \left(\frac{c_1}{\alpha}\right)^{p_1}\|h_A\|_{p_1}^{p_1}\right)\,d\alpha \\ &= 2^{q-1}qc_0^{q_0}\underbrace{\int_0^\infty \alpha^{q-1-q_0}\|g_A\|_{p_0}^{q_0}\,d\alpha}_{I} + 2^{q-1}qc_1^{q_1}\underbrace{\int_0^\infty \alpha^{q-1-q_1}\|h_A\|_{p_1}^{q_1}\,d\alpha}_{II}.\end{aligned}$$

We have

$$I \leq \int_0^\infty \alpha^{q-1-q_0}\, d\alpha p_0^{q_0/p_0} \left(\int_A^\infty \beta^{p_0-1} \lambda_f(\beta)\, d\beta \right)^{q_0/p_0}$$

The above inequality holds for every $A > 0$. Let $r > 0$ and choose $A = \alpha^r$ (it will turn out that r is the value we computed earlier). We will finish the proof next time. $\square$

5 The Marcinkiewicz Interpolation Theorem (cont.)

Today's lecture was given by a guest lecturer, Alpár Mészáros.

5.1 Continuation of the proof

Last time, we were proving the Marcinkiewicz interpolation theorem.

Theorem 5.1 (Marcinkiewicz interpolation theorem)**.** *Let $\mathcal{F}$ be the set of measurable functions on Y. Let $1 \leq p_0, p_1, q_0, q_1 \leq \infty$ be real numbers such that $p_0 \leq q_0$, $p_1 \leq q_1$, and $q_0 \neq q_1$. Let $t \in (0,1)$, and let p, q be defined as*

$$\frac{1}{p} = \frac{1-t}{p_0} + \frac{t}{p_1}, \qquad \frac{1}{q} = \frac{1-t}{q_0} + \frac{t}{q_1}.$$

Assume that $T : L^{p_0}(\mu) + L^{p_1}(\mu) \to \mathcal{F}$ be sublinear and of weak type (q_0, p_0) and (q_1, p_1) (there are $c_0, c_1 > 0$ such that if $q_0, q_1 \neq \infty$, $(\alpha^{q_0} \lambda_{T(f)})^{1/q_0} \leq c_0 \|f\|_{p_0}$ and $(\alpha^{q_1} \lambda_{T(f)})^{1/q_1} \leq c_1 \|f\|_{p_1}$). Then the following hold:

1. *T is strong type (p,q) (there exists $B_p > 0$ such that $\|Tf\|_q \leq B_p \|f\|_p$ for all $f \in L^p(\mu)$).*
2. *If $p_0 < \infty$, then $\lim_{p \to p_0} B_p |p_0 - p| < \infty$. If $p_1 < \infty$, then $\lim_{p \to p_1} B_p |p_1 - p| < \infty$. If $p_0 = \infty$, (B_p) remains bounded as $p \to p_0$. If $p_1 = \infty$, (B_p) remains bounded as $p \to p_1$.*

Proof. The general idea is the decompose the function f into two parts: for $A > 0$, cut off the function f if it exceeds A. So if $E(A) = \{x : |f(x)| > A\}$, we define $h_A = f \mathbb{1}_{X \setminus E(A)} + A \mathbb{1}_{E(A)}$ and $g_A = f - h_A$. First assume $q_0 \neq q_1$, and assume $q_0, q_1 < \infty$. Take q as in the theorem. If $f \in L^{p_0} + L^{p_1}$, then

$$\|Tf\|_q^q = q \int_0^\infty \alpha^{q-1} \lambda_{Tf}(\alpha)\, d\alpha$$

Since T is sublinear, we have $\lambda_{Tf}(2\alpha) \leq \lambda_{Tg_A}(\alpha) + \lambda_{Th_A}(\alpha)$ for all $\alpha, A > 0$ (independently of each other). We get, after a change of variables,

$$\|Tf\|_q^q \leq q 2^q \int_0^\infty \alpha^{q-1} \lambda_{Tf}(2\alpha)\, d\alpha \leq 2^q q \underbrace{\int_0^\infty \alpha^{q-1} \lambda_{Th_A}(\alpha)}_{=I_1} + \underbrace{\alpha^{q-1} \lambda_{Tg_A}(\alpha)\, d\alpha}_{=I_2}.$$

Look at I_2:

$$I_2 = 2^q q \int_0^\infty \alpha^{q-1} \frac{\alpha^{q_0}}{\alpha^{q_0}} \lambda_{Tg_A}(\alpha)\, d\alpha$$

$$
\begin{aligned}
&\le 2^q q \int_0^\infty \alpha^{q-q_0-1} [Tg_A]_{q_0}^{q_0}\, d\alpha \\
&\le 2^q q \int_0^\infty \alpha^{q-q_0-1} (c_0 \|g_A\|_{p_0})^{q_0}\, d\alpha \\
&= 2^q q C_0^{q_0} \int_0^\infty \alpha^{q-q_0-1} \|g_a\|_{p_0}^{q_0}\, \alpha.
\end{aligned}
$$

Now

$$
\begin{aligned}
\|g_A\|_{p_0}^{p_0} &= p_0 \int_0^\infty \alpha^{p_0-1} \lambda_{h_A}(\alpha)\, d\alpha \\
&= p_0 \int_0^\infty \alpha^{p_0-1} \lambda_f(\alpha + A)\, d\alpha \\
&= p_0 \int_A^\infty (\alpha - A)^{p_0-1} \lambda_f(\alpha)\, d\alpha \\
&\le p_0 \int_A^\infty \alpha^{p_0-1} \lambda_f(\alpha)\, d\alpha
\end{aligned}
$$

$$
\|h_A\|_{p_0}^{p_0} = p_0 \int_0^\infty \alpha^{q_0-1} \lambda_{h_A}(\alpha)\, d\alpha = p_0 \int_0^A \alpha^{p_0-1} \lambda_f(\alpha)\, d\alpha
$$

Combing back to $\|Tf\|_q^q$, we get

$$
\begin{aligned}
\|Tf\|_q^q &\le 2^q q C_0^{q_0} \int_0^\infty \alpha^{q-q_0-1} \|g_A\|_{p_0}^{q_0}\, d\alpha + 2^q q C_1^{q_1} \int_0^\infty \alpha^{q-q_1-1} \|h_A\|_{p_1}^{q_1} \\
&\le 2^q q C_0^{q_0} \int_0^\infty \alpha^{q-q_0-1} \left(p_0 \int_A^\infty \beta^{p_0-1} \lambda_f(\beta)\, d\beta \right)^{q_0/p_0} d\alpha \\
&\qquad + 2^q q C_1^{q_1} \int_0^\infty \alpha^{q-q_1-1} \left(p_1 \int_0^A \beta^{p_1-1} \lambda_f(\beta)\, d\beta \right)^{q_1/p_1} d\alpha \\
&= \sum_{j=0}^{1} 2^q q C_j^{q_j} p_j^{q_j/p_j} \int_0^\infty \left(\int_0^\infty \phi(\alpha, \beta)\, d\beta \right) d\alpha,
\end{aligned}
$$

where

$$
\phi(\alpha, \beta) := \mathbb{1}_j(\alpha, \beta) \beta^{p_j-1} \lambda_f(\beta) \alpha^{(q-q_j-1)p_j/q_j},
$$

$\mathbb{1}_0$ is the indicator of $\{(\alpha, \beta) : \beta > A\}$, and $\mathbb{1}_1$ is the indicator of $\{(\alpha, \beta) : \beta < A\}$.

In remains to study the terms separately with a special choice of A. Using Minkowski's inequality,

$$
\int_0^\infty \left(\int_0^\infty \phi_j(\alpha, \beta)\, d\beta \right)^{q_j/p_j} d\alpha \le \left(\int_0^\infty \left(\int_0^\infty \phi_j(\alpha, \beta)^{q_j/p_j}\, d\beta \right)^{p_j/q_j} d\alpha \right)^{q_j/p_j}
$$

Choose $\sigma > 0$ and set $A = \alpha^\sigma$. Then $\alpha \leq \beta^{1/\sigma}$. The inside of the above integral for $j = 0$ is (for a special choice of σ),

$$\begin{aligned}
\int_0^\infty \left(\int_0^{\beta^{1/\sigma}} \alpha^{q-q_0-1}\, d\alpha \right)^{p_0/q_0} \beta^{p_0-1} \lambda_f(\beta)\, d\beta &= \int_0^\infty \frac{1}{q-q_0} \left([\alpha]_0^{\beta^{1/\sigma}} \right)^{p_0/q_0} \beta^{p_0-1} \lambda_f(\beta)\, d\beta \\
&= (q-q_0)^{-p_0/q_0} \int_0^\infty \beta^{p_0-1+(q-q_0)/\sigma} \lambda_f(\beta)\, d\beta \\
&= (q-q_0)^{-p_0/q_0} \int_0^\infty \beta^{p-1} \lambda_f(\beta)\, d\beta \\
&= (q-q_0)^{-p_0/q_0} p^{-1} \|f\|_p^p.
\end{aligned}$$

The other term is similar. We will finish the proof next time. $\square$

6 The Marcinkiewicz Interpolation Theorem (cont.)[2]

Today's lecture was given by a guest lecturer, Alpár Mészáros.

6.1 Conclusion of the proof

Last time, we were proving the following theorem.

Theorem 6.1 (Marcinkiewicz[1] interpolation theorem)**.** *Let $\mathcal{F}$ be the set of measurable functions on Y. Let $1 \le p_0, p_1, q_0, q_1 \le \infty$ be real numbers such that $p_0 \le q_0$, $p_1 \le q_1$, and $q_0 \neq q_1$. Let $t \in (0,1)$, and let p, q be defined as*

$$\frac{1}{p} = \frac{1-t}{p_0} + \frac{t}{p_1}, \qquad \frac{1}{q} = \frac{1-t}{q_0} + \frac{t}{q_1}.$$

Assume that $T : L^{p_0}(\mu) + L^{p_1}(\mu) \to \mathcal{F}$ be sublinear and of weak type (q_0, p_0) and (q_1, p_1) (there are $c_0, c_1 > 0$ such that if $q_0, q_1 \neq \infty$, $(\alpha^{q_0}\lambda_{T(f)})^{1/q_0} \le c_0\|f\|_{p_0}$ and $(\alpha^{q_1}\lambda_{T(f)})^{1/q_1} \le c_1\|f\|_{p_1}$). Then the following hold:

1. *T is strong type (p,q) (there exists $B_p > 0$ such that $\|Tf\|_q \le B_p\|f\|_p$ for all $f \in L^p(\mu)$).*

2. *If $p_0 < \infty$, then $\lim_{p\to p_0} B_p|p_0 - p| < \infty$. If $p_1 < \infty$, then $\lim_{p \to p_1} B_p|p_1 - p| < \infty$. If $p_0 = \infty$, (B_p) remains bounded as $p \to p_0$. If $p_1 = \infty$, (B_p) remains bounded as $p \to p_1$.*

Proof. Without loss of generality we can assume $p_0 < p_1$. We showed that

$$\|Tf\|_q^q \le \sum_{j=0}^{1} 2^q q C_j^{q_j} p^{q_j/p_j} \int_0^\infty \left(\int_0^\infty \phi_j(\alpha,\beta)\,d\beta\right)^{q_j/p_j} d\alpha.$$

Here,

$$\phi_j(\alpha,\beta) := \mathbb{1}_j(\alpha,\beta)\beta^{p_j-1}\lambda_f(\beta)\alpha^{(q-q_j-1)p_j/q_j},$$

where $\mathbb{1}_0$ is the indicator of $\{(\alpha,\beta) : \beta > A\}$ and $\mathbb{1}_1$ is the indicator of $\{(\alpha,\beta) : \beta < A\}$. We want to set $A = \alpha^\sigma$ for some good choice of σ. Look at the term with ϕ_0.

Case 1: $\sigma > 0$: If $\beta > \alpha^\sigma$, then $\alpha < \beta^{1/\sigma}$. After Minkowski's inequality,

$$\begin{aligned}\int_0^\infty &\left(\int_0^{\beta^{1/\sigma}} \alpha^{q-q_0-1}\,d\alpha\right)^{p_0/q_0} \beta^{p_0-1}\lambda_f(\beta)\,d\beta \\ &= \int_0^\infty \left(\frac{1}{q-q_0}\right)^{p_0/q_0} \beta^{(q-q_0)p_0/(q_0\sigma)}\beta^{p_0-1}\lambda_f(\beta)\,d\beta\end{aligned}$$

[1] Marcinkiewicz was a Polish mathematician who died during WWII. Zygmund discovered afterwards that he proved this result and gave credit to Marcinkiewicz.

Now pick

$$\sigma = \frac{p_0}{q_0}\frac{q_0 - q}{p_0 - p} > 0.$$

Since we want this to be positive, we need to assume that $q_0 < q_1$. The previous quantity becomes

$$\left(\frac{1}{q - q_0}\right)^{p_0/q_0} p^{-1}\|f\|_p^p.$$

If $q_0 > q_1$, then $\sigma < 0$. So $\beta > \alpha \implies \alpha > \beta^{1/\sigma}$. Then what changes is the integral becomes an integral $\int_0^\infty \int_{\beta^{1/\sigma}}^\infty$. We get

$$\int_0^\infty \frac{1}{q - q_0}\left([\alpha^{q-q_0}]_{\beta^{1/\sigma}}^\infty\right)^{p_0/q_0} d\beta = \left(\frac{1}{q_0 - q}\right)^{p_0/q_0} p^{-1}\|f\|_p^p.$$

For the term involving ϕ_1, the computation is very similar with (p_1, q_1) instead of (p_0, q_0). The key property here is that

$$\sigma = \frac{p_0}{q_0}\frac{q_0 - q}{p_0 - p} = \frac{p_1}{q_1}\frac{q_1 - q}{p_1 - p},$$

which follows from the construction of p, q.

Remaining case 1: Assume that $p_1 = q_1 = \infty$. Then $\|Tf\|_\infty \le C_1\|f\|_\infty$ (because (∞,∞)-weak means (∞,∞)-strong). We have $\|Th_A\|_\infty \le C_1\|h_A\|_\infty$. We want to choose A in a way that ϕ_1 becomes 0. We claim that $A = \alpha/C_1$ works. In this case, $\beta < \alpha/C_1$. We get

$$\|Th_A\|_\infty \le C_1\|h_A\|_\infty \le C_1 A = C_1\frac{\alpha}{C_1} = \alpha.$$

We have

$$\mathbb{1}_{\{\beta<\alpha/C_1\}}\lambda_f(\beta) = \mathbb{1}_{\{\beta<\alpha/C_1\}}\lambda_f h_A(\beta) = 0,$$

so ϕ_1 does not give a contribution. Do the same computation with ϕ_0, replacing α^σ with α/C_1.

Remaining case 2: Assume $p_0 < p_1 < \infty$ and $q_0 < q_1 = \infty$. Choose A in a way such that $\lambda_{Th_A}(\beta) = 0$ ($\|Th_A\|_\infty \le C_1\|f\|_{p_1}$). If we choose $A = (\alpha/d)^\sigma$, where $\sigma = p_1/(p_1 - p)$ and $d = C_1[p_1\|f\|_p^p/p]^{1/p_1}$, we get

$$\|Th_A\| \le \alpha.$$

Remaining case 3: If $p_0 < p_1 < \infty$ and $q_1 < q_0 = \infty$, we want that $\lambda_{Tg_A}(\alpha) = 0$. In this case, choose A such that $A = (\alpha/d)^\sigma$.

We have obtained that

$$\|Tf\|_q^q \le \text{constant}\,\|f\|_p^p.$$

Define B_p such that $\sup\{\|Tf\|_q : \|f\|_p = 1\} \le B_p$. You can write down the constant explicitly in all cases. $\square$

6.2 L^p-estimates for the Hardy-Littlewood Maximal function

For $f \in L^1_{\text{loc}}$, let

$$(Hf)(x) = \sup_{r>0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y)|\, dy$$

be the Hardy-Littlewood maximal function. Then H is sublinear. H is (∞, ∞)-strong type. We can show that H is $(1,1)$-weak type. By the Marcinkiewicz interpolation theorem, we get that

$$\|Hf\|_p \leq C(n) \frac{p}{p-1} \|f\|_p$$

for any $p \in (1, \infty]$.

However, H is not $(1,1)$-strong type. Come up with an example as an exercise.

7 Bounds on Kernel Operators

7.1 Strengthening of a previous theorem

We will prove a stronger version of the following theorem.

Theorem 7.1. *Let* $(X, \mathcal{M}, \mu)$ *and* $(\mathcal{Y}, \mathcal{N}, \nu)$ *be* σ*-finite measure spaces. Let* $K : X \times Y \to \mathbb{R}$ *be* $\mathcal{M} \otimes \mathcal{N}$*-measurable, and let* $\mathcal{F}$ *be the set of* $f : Y \to \mathbb{R}$ *measurable functions such that* $K(x, \cdot)f \in L^1$ *for* μ*-a.e.* $x \in X$*. For* $f \in \mathcal{F}$*, define*

$$Tf(x) = \int_Y K(x,y) f(y)\, d\nu(y).$$

Assume there exists $C > 0$ *such that*

$$\int_Y |K(x,y)|\, d\nu(y) \le C$$

for μ*-a.e.* $x \in X$ *and*

$$\int_X |K(x,y)|\, d\mu(x)$$

for ν*-a.e.* $y \in Y$*. The the following conclusions hold:*

1. *For any* $1 \le p < \infty$*,* $L^p(\nu) \subseteq \mathcal{F}$*.*
2. *There exists* C_p *such that* $\|Tf\| \le C_p \|f\|_p$ *if* $f \in L^p(\nu)$*.*

Recall that if $A > 0$, then

$$\phi_A(z) = \begin{cases} z & |z| < A \\ \frac{z}{|z|} A & |z| \ge A \end{cases}$$

is a function in $C(\mathbb{C}, \mathbb{C})$, and $\phi_A|_{\mathbb{R}} \in C(\mathbb{R}, \mathbb{R})$. Observe that

$$z - \phi_A(z) = \begin{cases} 0 & |z| < A \\ \frac{z}{|z|}(|z| - A) & |z| \ge A. \end{cases}$$

We shall use the notation

$$K_1 = K_1^A = K - \phi_A(K), \qquad K_2 = K_2^A = \phi_A(K).$$

Denote as T_i $(i = 1, 2)$ the operators associated to K_i $(i = 1, 2)$.

Theorem 7.2. *Let* $1 \le p < \infty$ *and* $c > 0$*. Assume that* $[K(x, \cdot)]_q \le C$ *for* μ*-a.e.* $x \in X$ *and* $[K(\cdot, w)]_w \le C$ *for* ν*-a.e.* $y \in Y$*.*

1. *If $1 \leq p < \infty$, $L^p(\nu) \subseteq \mathcal{F}$.*
2. *If $1 < p < r < \infty$, then there exist $B_1 > 0$ and $B_p > 0$ such that $[Tf]_q \leq B_1\|f\|_1$ and $\|Tf\|_r \leq CB_p\|f\|_p$, which means T is weak type $(1,q)$ and strong type (p,r), provided that $1/r + 1 = 1/p + 1/q$.*

Proof. Let $f \in L^p(\nu)$; we want $f \in \mathcal{F}$. If $f = 0$, we are done. If $f \neq 0$, it suffices to show that $f/\|f\|_p \in \mathcal{F}$. So we need only show that if $\|f\|_p = 1$, then $f \in \mathcal{F}$. For the second conclusion, let $f \in L^p$. If $f = 0$, then the conclusion holds. If $f \neq 0$, then we can again reduce to the case $\|f\|_p = 1$ by passing to $f/\|f\|_p$. So it suffices to prove both parts when $\|f\|_p = 1$.

Let $f \in L^p(\nu)$ be such that $\|f\| = 1$. Let q' be the dual conjugate of q, and let p' be the dual conjugate of q. We have $1/r = 1/p + 1/q - 1 = 1/p - 1/q'$, and similarly, $1/r = -1/q' + 1/q$. Since $r > 0$, $1/p > 1/q'$, and $1/q > 1/p'$. So $q' > p$, and $p' > q$. We have

$$\alpha^q \lambda_{K(x,\cdot)}(\alpha) \leq C, \qquad \alpha^q \lambda_{K(\cdot,y)}(\alpha) \leq C.$$

To show that $|K(x,\cdot)f| \in L^1(\nu)$, we are going to show that $|K_i(x,\cdot)f| \in L1(\nu)$ for $i = 1, 2$. We have

$$\begin{aligned}\int_Y |K_1(x,y)|\, d\nu(y) &= \int_0^\infty \lambda_{K_1(x,\cdot)}(\alpha)\, d\alpha = \int_0^\infty \lambda_{K(x,\cdot)}(\alpha + A)\, d\alpha \\ &= \int_A^\infty \lambda_{K(x,\cdot)}(\alpha)\, d\alpha \leq C\int_A^\infty \alpha^{-q}\, d\alpha = C\frac{A^{1-q}}{q-1}.\end{aligned}$$

The similar identity holds for $\int_X |K_1(x,y)|\, d\mu(x)$, so we have

$$\int_Y |K(x,y)|\, d\nu(y), \int_X |K(x,y)|\, d\mu(x) \leq C\frac{A^{1-q}}{q-1}.$$

We have

$$\begin{aligned}\int_Y |K_2(x,y)|^{p'}\, d\nu(y) &= p'\int_0^\infty \lambda_{K_2(x,\cdot)}(\alpha)\alpha^{p'-1}\, d\alpha = p'\int_0^A \lambda_{K(x,\cdot)}(\alpha)\alpha^{p'-1} \\ &\leq p'\int_0^A C\alpha^{p'-1-q}\, d\alpha = C\frac{p'}{p'-q}A^{p'-q}.\end{aligned}$$

By symmetry, we get that

$$\int_Y |K_2(x,y)|^{p'}\, d\nu(y), \int_X |K_2(x,y)|^{p'}\, d\mu(x) \leq C\frac{p'}{p'-q}A^{p'-q}.$$

Apply Hölder's inequality to conclude that

$$\int_Y |K_2(x,y)f(x)|\, d\nu(y) \leq \left(\int_Y |K_2(x,y)|^{p'}\, d\nu(y)\right)^{1/p'} \|f\|_p \leq \left(C\frac{p'}{p'-q}\right)^{1/p'} A^{1-q/p'}$$

So $K_2(x,\cdot)f \in L^1(\nu)$. Using the previous theorem, we conclude that $K_1(x,\cdot)f \in L^1(\nu)$. In conclusion, $K(x,\cdot)f \in L^1(\nu)$, which implies that $L^p(\nu) \subseteq \mathcal{F}$.

Choosing an appropriate A: By our inequality,

$$\|T_2 f\| \le \left(C\frac{p'}{p'-q}\right)^{1/p'} A^{1-q/p'}.$$

Choose A such that

$$\left(C\frac{p'}{p'-q}\right)^{1/p'} A^{q/r} = \left(C\frac{p'}{p'-q}\right)^{1/p'} A^{1-q/p'} = \frac{\alpha}{2}.$$

That is, we choose

$$A = \left[\left(C\frac{p'}{p'-q}\right)^{1/p'} A^{q/r}\right]^{r/q}.$$

By assumption, $\|T_2 f\| \le \alpha/2$, and so $\lambda_{T_2 f}(\alpha/2) = 0$. $\square$

Next time, we will finish the proof.

8 Bounds on Integral Operators (cont.)

8.1 Proof of the weak and strong type properties

Last time, we were proving the following theorem:

Theorem 8.1. *Let $1 \leq p < \infty$ and $c > 0$. Assume that $[K(x,\cdot)]_q \leq C$ for μ-a.e. $x \in X$ and $[K(\cdot,w)]_w \leq C$ for ν-a.e. $y \in Y$.*

1. *If $1 \leq p < \infty$, $L^p(\nu) \subseteq \mathcal{F}$.*
2. *If $1 < p < r < \infty$, then there exist $B_1 > 0$ and $B_p > 0$ such that $[Tf]_q \leq B_1 \|f\|_1$ and $\|Tf\|_r \leq CB_p\|f\|_p$, which means T is weak type $(1,q)$ and strong type (p,r), provided that $1/r + 1 = 1/p + 1/q$.*

Proof. It remains to show the second conclusion. We have fixed f such that $\|f\|_p = 1$. We have already obtained the following useful identities:

$$\int_X |K_1(x,y)|\, d\nu(y), \int_K |K_1(x,y)|\, d\nu(x) \leq C\frac{A^{1-q}}{q-1}$$

$$\|T_2 f\| \leq A^{q/r}\left(\frac{cr}{q}\right)^{1/p}, \qquad \frac{1}{p} + \frac{1}{p'} = 1.$$

We chose A such that $A^{q/r}(cr/q)^{1/p'} = \alpha/2$. These give us

$$\lambda_{T_2 f}(\alpha/2) = 0.$$

So

$$\lambda_{Tf}(\alpha) \leq \lambda_{T_1 f}(\alpha/2) + \lambda_{T_2 f}(\alpha/2) = \lambda_{T_1 f}(\alpha/2).$$

Now apply the following observation to $h = T_1 f$:

$$\int |h|^p\, d\nu \geq \int_{\{|h| > \alpha/2\}} |h|^p\, d\nu \geq \left(\frac{\alpha}{2}\right)^p \lambda_h(\alpha/2) \implies \lambda_h(\alpha/2) \leq \left(\frac{\alpha}{2}\right)^{-p} \|h\|_p^p.$$

We get

$$\begin{aligned}
\lambda_{Tf}(\alpha) &\leq \left(\frac{\alpha}{2}\right)^{-p} \|T_1 f\|_p^p \\
&\leq \left(\frac{\alpha}{2}\right)^{-p} \left(c\frac{A^{1-q}}{q-1}\right)^p \\
&= \left(\frac{\alpha}{2}\right)^{-p} \left(\frac{c}{q-1}\alpha^{r/q}\left[\frac{1}{2}\left(\frac{q}{cr}\right)^{1/p'}\right]^{r/q}\right)^{(1-q)p} \\
&= \alpha^{-p+r/q(1-q)p} C(q,p).
\end{aligned}$$

Now we note that

$$-p + r/q(1-q)p = p((1/q-1)-1) = p(r(1/r-1/p)-1) = -r/p.$$

So, by homogeneity,

$$\alpha^r \lambda_{Tf}(\alpha) \leq C(q,p)\|f\|_p^r.$$

In particular, when $p=1$, then $r=q$, and we get that

$$\alpha^q \lambda_{Tf}(\alpha) \leq C(q,1)\|f\|_q^q.$$

That is, T is weak type $(1,q)$.

We next need to find (p_1, r_1) such that T is weak type (p_1, r_1), where $q \geq 1$ and $p_1 \leq r_1$. Choose $p_1 \in (p, \infty)$ close enough to p. Let $t \in (0,1)$ be such that

$$\frac{1}{p} = \frac{1-t}{1} + \frac{t}{p_1}.$$

Define r_1 by

$$\frac{1}{r} = \frac{1-t}{q} + \frac{t}{r_1}.$$

Since p is close to p_1, r is close to r_1. By the definition of r_1,$r_1 < r$. We have

$$\alpha^{r_1} \lambda_{Tf}(\alpha) \leq C(q,p_1)\|f\|_{p_1}^{r_1}.$$

This means that T is weak type (p_1, r_1). Since T is also weak type $(1,q)$ the Marcinkiewicz interpolation theorem gives us that T is strong type (p,r). $\square$

8.2 Preliminaries for Fourier analysis

Notation: We will assume that $n \geq 1$ is a natural number. If $x = (x_1, \ldots, x_n), (y_1, \ldots, y_n) \in \mathbb{R}^n$, then

$$x \cdot y = \sum_{i=1}^{n} x_i y_i, \qquad \|x\|^2 = x \cdot x.$$

If $\alpha \in \mathbb{N}^n$, then

$$|\alpha| = \sum_{i=1}^{n} \alpha_i, \qquad \alpha! = \prod_{i=1}^{n} (\alpha_i!).$$

We will also write

$$x^\alpha = (x_1^{\alpha_1}, \ldots, x_n^{\alpha_n}), \qquad \partial_x^\alpha = \frac{\partial^{|\alpha|}}{\partial x^\alpha} = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_1^{\alpha_n}}.$$

With this notation, the Taylor expansion is

$$f(x) = \sum_{|\alpha| \le k} \frac{1}{\alpha!} \frac{\partial^{|\alpha|} f}{\partial x^\alpha}(x_0)(x - x_0)^\alpha + R_k(x), \qquad \text{where } \lim_{x \ tox_0} \frac{R_k(x)}{|x - x_0|^k} = 0.$$

Define

$$\eta(t) = \begin{cases} e^{-1/t} & t > 0 \\ 0 & t \le 0. \end{cases}$$

We have $\eta \in C^\infty(\mathbb{R})$, as

$$\frac{x^n}{e^x} \xrightarrow{x \to \infty} 0$$

for each n. By induction, we can show that $\eta^{(k)}(0) = 0$ for all $k \ge 1$.

For $x \in \mathbb{R}^n$, set

$$\rho(x) = \eta(1 - \|x\|^2) = \begin{cases} e^{1/(\|x\|^2 - 1)} & \|x\| < 1 \\ 0 & \|x\| > 1. \end{cases}$$

Then $\operatorname{supp}(\rho) = \overline{B_1(0)}$, $\rho \in C^\infty$, $\rho > 0$, and $\rho(-x) = x$.

9 The Schwarz Space

9.1 Topology of the Schwarz space

Definition 9.1. Given $N \geq 0$ and $\alpha \in \mathbb{N}^n$ ($\mathbb{N} = \{0, 1, 2, \dots\}$), we define the seminorm of $f \in C^\infty(\mathbb{R}^n)$

$$\|f\|_{(N,\alpha)} := \sup_x (1 + |x|)^N |\partial^\alpha f(x)|.$$

The **Schwarz space** is $\mathcal{S} = \{f \in C^\infty(\mathbb{R}^n) : \|f\|_{(N,\alpha)} < \infty \,\forall N \in \mathbb{N}, \alpha \in \mathbb{N}^n\}$.

Example 9.1. If $f \in C^\infty(\mathbb{R}^n)$ with compact support, then $f \in \mathcal{S}$.

Example 9.2. $|\partial^\alpha (e^{-|x|^2})| \leq c(1 + |x|^{2|\alpha|})e^{-|x|^2}$.

$\mathcal{S}$ is endowed with a topology induced by the seminorm as follows: $(f_k)_k \subseteq \mathcal{S}$ converges to $f \in \mathcal{S}$ iff

$$\lim_{k\to\infty} \|f_k - f\|_{(N,\alpha)} = 0$$

for all $N \in \mathbb{N}$ and $\alpha \in \mathbb{N}^n$. Recall that a Freéchet is a complete, Hausdorff, topological vector space whose topology is induced by a countable family of seminorms.

Proposition 9.1. *$\mathcal{S}$ is a Fréchet space.*

Proof. Hausdorff: Given $f \in \mathcal{S}$ and $\varepsilon > 0$, $U^\varepsilon_{(N,\alpha)} = \{g \in \mathcal{S} : \|f - g\|_{(N,\alpha)} < \varepsilon\}$ are the open sets that generate the topology of $\mathcal{S}$. Let $f_1, f_2 \in \mathcal{S}$ be distinct. Let $x_0 \in \mathbb{R}^n$ be such that $4\delta := |f_1(x_0) - f_2(x_0)| > 0$. Since $|f_1 - f_2|$ is continuous, there exists an open neighborhood O of x_0 such that $|f_1(x) - f_2(x)| \geq 3\delta$ for all $x \in O$. We have $U^\delta_{(0,0)}(f_1) \cap U^\delta_{(0,0)}(f_2) = \varnothing$. This proves that $\mathcal{S}$ is a Hausdorff space.

Completeness: Let $(f_k)_k \subseteq \mathcal{S}$ be a Cauchy sequence: $\lim_{k,\ell\to\infty} \|f_k - f_\ell\|_{(N,\alpha)} = 0$ for all $N \in \mathbb{N}, \alpha \in \mathbb{N}^n$. Taking $N = 0$ for each α, we obtain that $(\partial^\alpha f_k)_k$ is a Cauchy sequence for the uniform norm, and so $(\partial^\alpha f_k)_k$ converges uniformly to some $g_\alpha \in C(\mathbb{R}^n)$. We claim that $\sup_x (1 + |x|)^N g_\alpha(x) < \infty$. We have $(1|x|)^n |\partial^\alpha f_k - \partial^\alpha f_\ell| \leq \varepsilon$ for large k, ℓ. Letting $\ell \to \infty$, we get $1|x|)^n |\partial^\alpha f_k - g_\alpha|$ for large k. Then

$$(1 + |x|^N |g_\alpha| \leq \underbrace{(1 + |x|)^N |g_\alpha(x) - \partial^\alpha f_k(x)|}_{\leq\varepsilon} + (1 + |x|)^N |\partial^\alpha f_k(x)| < \infty.$$

It remains to show that $g_0 \in C^\infty(\mathbb{R}^n)$ and $\partial^\alpha g_0 = g_\alpha$. By Taylor's expansion,

$$f_k(x + h) = f_k(x) - \nabla f_k(x)h = \int_0^1 \int_0^1 (\nabla^2 f_k(x + tsh))h \cdot h)\, ds\, dt.$$

Thus,

$$|f_k(x + h) - f_k(x) - h \cdot \nabla f_j(x)| \leq \frac{|h|^2}{2} M, \qquad M = \sup_k \sup_{|\alpha|=2} \|f_k\|_{(0,\infty)}.$$

Letting $k \to \infty$, we obtain

$$\left| g_0(x+h) - g_0(x) - \sum_{i=1}^{n} g_{(0,\cdots,0,1,0,\ldots,0)}(x) h_i \right| \leq \frac{M}{2} \|h\|^2.$$

Since $g_{(0,\cdots,0,1,0,\ldots,0)}(x)$ is continuous, we conclude that g_0 is differentiable at x and that $\frac{\partial}{\partial x_i} g_0(x) = g_{(0,\ldots,0,1,0,\ldots,0)}(x)$. Increasing the rank of the expansion, we obtain the desired result. So $g_\alpha = \partial^\alpha f$. $\square$

9.2 Equivalent characterizations of functions in the Schwarz space

Proposition 9.2. *Let $f \in C^\infty(\mathbb{R}^n)$. The following are equivalent:*

1. *$f \in \mathcal{S}$.*
2. *$x^\beta \partial^\alpha f$ is bounded for any $\beta, \alpha \in \mathbb{N}^n$.*
3. *$\partial^\alpha (x^\beta f)$ is bounded for any $\beta, \alpha \in \mathbb{N}^n$.*

Proof. (1) $\implies$ (2): Let $\alpha, \beta \in \mathbb{N}^n$. Then

$$|x^\beta||\partial^\alpha f(x)| \leq (1+|x|)^{|\beta|} |\partial^\alpha f(x)| \leq \|f\|_{(|\beta|,\alpha)}.$$

(2) $\implies$ (3): We have

$$\partial^\alpha (x^\beta f) = \sum_{a \in A, b \in B} x^a \partial^b f,$$

where A and B are finite sets determined by α, β. Thus,

$$|\partial^\alpha (x^\beta f(x))| \leq \sum_{a \in A, b \in B} \|x^\alpha \partial^b \beta\| < \infty.$$

(3) $\implies$ (1): We have $\|\partial^\alpha f\|_\infty < \infty$ for all $\alpha \in \mathbb{N}^n$. It remains to show that $\|(1+|x|)^N \partial^\alpha f(x)\|_\infty < \infty$. Fix an integer $N \geq 1$. Then

$$\delta_N := \min\{\sum_{i=1}^{n} |x_i|^N : \|x\| = 1\} > 0.$$

Hence,

$$\delta_N \leq \sum_{i=1}^{n} \left| \frac{x_i}{\|x\|} \right|^N = \frac{1}{\|x\|^N} \sum_{i=1}^{N} |x_i|^N.$$

So

$$\|x\|^N \leq \frac{1}{\delta_N} \sum_{i=1}^{n} |x_i|^N.$$

It remains to show that $\||x_i|^N \partial^\alpha f\|_\infty < \infty$. We have for $N = 1$ that

$$\partial_{x_j}(x_i \partial^\alpha f) = \delta_{i,j} \partial^\alpha f + x_i \partial_{x_j} \partial^\alpha f,$$

so

$$\|x_i \partial_{x_j} \partial^\alpha f\| \leq \|\partial_{x_j}(x_i \partial^\alpha f)\|_\infty + \|\partial \alpha^\alpha f\|_\infty^j.$$

Repeat the process for $N = 2, 3, \ldots$. □

10 Translation and Convolution

10.1 Translations of functions

Definition 10.1. Given $f : \mathbb{R}^n \to \mathbb{R}$ and $y \in \mathbb{R}^n$, define the **translation** $\tau_y f : \mathbb{R}^n \to \mathbb{R}^n$ by

$$(\tau_y f)(x) = f(x - y).$$

Remark 10.1. If $1 \leq p \leq \infty$ and $y \in \mathbb{R}$, $\tau_y : L^p \to L^p$ is an isometry.

Remark 10.2. If $f \in \mathbb{R}^n \to \mathbb{R}$, then f is uniformly continuous if and only if the limit $\lim_{y\to 0} \|\tau_y f - f\|_u = 0$. Indeed,

$$\sup_{\|y\|\leq\delta} \|\tau_y - f\|_u = \sup_{\|z-y\|\leq\delta} |f(y) - f(z)|.$$

Remark 10.3. If $f : \mathbb{R}^n \to \mathbb{R}$ is supported by the ball $B_R(0)$, then

$$\|\tau_y f - f\|_p^p \leq |B_{R+1}(0)|^{1/p} \|\tau_y - f\|_\infty$$

whenever $\|y\| \leq 1$. Indeed,

$$\int_{\mathbb{R}^n} |f(x) - f(x-y)|^p \, dx = \int_{B_{R+1}(0)} |f(x) - f(x-y)|^p \, dx.$$

Let $C_c(\mathbb{R}^n)$ be the set of continuous functions $\mathbb{R}^n \to \mathbb{R}$ with compact support.

Lemma 10.1. *If $g \in C_c(\mathbb{R}^n)$, then g is uniformly continuous.*

Proof. Let $B_{R-1}(0)$ with $R > 1$ be a ball containing the support of $g \in C_c(\mathbb{R}^n)$. Then g is uniformly continuous on $\overline{B_R(0)}$. Set

$$\delta(r) = \sup_{\substack{\|x-y\|<r \\ \|x\|,\|y\|\leq R+1}} |g(x) - g(y)|.$$

We have

$$|g(x) - g(y)| \leq \begin{cases} 0 & \|y\| \geq R, \|x\| \geq R+1 \\ |g(x) - g(x_0)| & \|y\| < R, \|x\| \geq R+1 \\ 0 & \|x\| \geq R, \|y\| \geq R+1 \\ |g(x) - g(z_0)| & \|x\| \leq R, \|y\| \geq R+1. \end{cases}$$

Consequently,

$$\sup_{\|x-y\|\leq r} |g(x) - g(y)| \leq \delta(r).$$

So g is uniformly continuous. ☐

Proposition 10.1. *If $1 \le p < \infty$, then τ_y converges pointwise to the identity map in L^p:*

$$\lim_{y \to 0} \|\tau_y f - f\|_p = 0.$$

Proof. Let $f \in L^p$. For any $g \in C_c(\mathbb{R}^n)$ and $y \in \mathbb{R}^n$, we have

$$\begin{aligned} \|\tau_y f - f\|_p &\le \|\tau_y f - \tau_y g\|_p + \|\tau_y g - g\|_p + \|g - f\|_p \\ &= 2\|f - g\| + \|\tau_y g - f\|_p \\ &\le 2\|f - g\| + |B_R|^{1/p} \|\tau_y g - f\|_u, \end{aligned}$$

where $B_{R-1}(0)$ is a ball containing the support of g. Since g is uniformly continuous, we conclude

$$\limsup_{y \to 0} \|\tau_y f - f\|_p \le 2\|f - g\|_p.$$

Since $C_c(\mathbb{R}^n)$ is dense in L^p,

$$\limsup_{y \to 0} \|\tau_y f - f\|_p = 0.$$ ☐

10.2 Convolution

Definition 10.2. Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be measurable, and let $x \in \mathbb{R}^n$ be such that $y \mapsto \tau_y f g$ is integrable. Then we define the **convolution** of f and g as

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x - y) g(y)\, dy = \int \tau_y f(x) g(y)\, dy.$$

Definition 10.3. The n**-torus** is $\mathbb{T}^n := \mathbb{R}^n / \mathbb{Z}^n$.

If $x \in \mathbb{R}^n$, the equivalence class of x in $\mathbb{T}^n$ is $x + \mathbb{Z}^n = \hat{x}$. The metric on $\mathbb{T}^n$ is

$$\|\hat{x} - \hat{y}\|_{\mathbb{T}^n} = \inf_{z \in \mathbb{Z}^n} |x - y - z|.$$

There is a bijection between $\mathbb{T}^n$ and $Q_n = [-1/2, 1/2)^n$. Consequently, there is a bijection between $\mathbb{T}^n$ and $\tilde{Q}_n = \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n : |z_i| = 1\ \forall i\}$. Since $\tilde{Q}_n$ is compact, we conclude that $\mathbb{T}^n$ is a compact set.

Proposition 10.2. *If $x \in \mathbb{R}^n$, $f, g : \mathbb{R}^n \to \mathbb{R}$ are masurable, and $y \mapsto \tau_y f(x) g(y)$ is integrable, then*

$$(f * g)(x) = (g * f)(x).$$

Proof. Use the change of variables $z = x - y$:

$$\begin{aligned} (f * g)(x) &= \int_{\mathbb{R}^n} f(x - y) g(y)\, dy \\ &= \int_{\mathbb{R}^n} f(z) g(x - z)\, dz \\ &= (g * f)(x). \end{aligned}$$ ☐

11 Properties of Convolution and Young's Inequality

11.1 Properties of convolution

Proposition 11.1. *The convolution satisfies the following properties:*

1. $f * g = g * f$
2. $(f * g) * h = f * (g * h)$
3. *If* $z \in \mathbb{R}^n$, $\tau_x(f * g) = (\tau_z f) * g = f * (\tau_z g)$.
4. *If* $A = \{x + y : x \in \operatorname{supp}(f), y \in \operatorname{supp}(g)\}$, *then* $\operatorname{supp}(f * g) \subseteq \overline{A}$.

Proof. Let $f, g : \mathbb{R}^n \to \mathbb{R}$.

1. We have already proved this.
2. Let $x \in \mathbb{R}^n$. We have

$$\begin{aligned}(f * g) * h(x) &= \int_{\mathbb{R}^n} (f * g)(x - y)h(y)\, dy \\ &= \int_{\mathbb{R}^n \times \mathbb{R}^n} f(x - y - z)g(z)h(y)\, dy\, dz\end{aligned}$$

   But

$$f * (g * h)(x) = \int_{\mathbb{R}}^{n} f(x - u)g * h(u)\, du = \int_{\mathbb{R}^n \times \mathbb{R}^n} f(x - u)g(u - v)h(v).$$

   So set $u = y$ and $u - v = z$. Then

$$x - y - z = x - v - (u - v) = x - u,$$

   so the two expressions are equal.

3. We have

$$\begin{aligned}\tau_z(f * g)(z) &= f * g(x - z) \\ &= \int_{\mathbb{R}^n} f(x - z - y)g9y)\, dy \\ &= \int_{\mathbb{R}^n} (\tau_z f)(x - y)g(y)\, dy \\ &= (\tau_z f) * g(x).\end{aligned}$$

   Since $f * g = g * f$, we conclude that

$$\tau_z(f * g) = \tau_z(g * f) = (\tau_z g) * f = f * (\tau_z g).$$

4. Assume $x \notin A$. Observe that

$$f(x-y)g(y) = \begin{cases} 0 & y \notin \operatorname{supp}(g) \\ 0 & y \notin \operatorname{supp}(g). \end{cases}$$

Hence, $f * g(x) = 0$. Then $A^c \subseteq \{f * g = 0\}$, so $\{f * g \neq 0\} \subseteq A$, which makes $\overline{\{f * g \neq 0\}} \subseteq \overline{A}$. $\square$

11.2 Young's inequality

Our goal is that if $1 \leq p, q, r < \infty$ and $r^{-1} + 1 = p^{-1} + q^{-1}$, then

$$\|f * g\|_r \leq \|f\|_p \|g\|_q.$$

It is important to note here that this bound is independent of the dimension.

Theorem 11.1 (Young's inequality)**.** *Let $1 \leq q \leq \infty$, let $f \in L^1$, and let $g \in L^q$. For a.e. $x \in \mathbb{R}^n$, $f * g(x)$ exists, and*

$$\|f * g\|_q \leq \|f\|_1 \|g\|_q.$$

Proof. Assume $q < \infty$.

$$\begin{aligned} \|f * g\|_q &= \left(\int_{\mathbb{R}^n} |f * g(x)|^q \, dx \right)^{1/q} \\ &= \left(\int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} f(y) g(x-y) \, dy \right|^q \, dx \right)^{1/q} \end{aligned}$$

Use Minkowski's inequality.

$$\begin{aligned} &\leq \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |f(y) g(x-y)|^q \, dx \right)^{1/q} \, dy \\ &= \int_{\mathbb{R}^n} |f(y)| \left(\int_{\mathbb{R}^n} |g(x-y)|^q \, dx \right)^{1/q} \, dy \end{aligned}$$

Set $z = x - y$.

$$= \int_{\mathbb{R}^n} |f(y)| \|g\|_q \, dy = \|f\|_1 \|g\|_q.$$

If $q = \infty$, the proof is simpler. $\square$

Definition 11.1. $C_0(\mathbb{R}^n) = \{f \in C(\mathbb{R}^n) : \{|f| \geq \varepsilon\}$ is compact $\forall \varepsilon > 0\}$ is the set of functions that **vanish at** ∞.

Remark 11.1. As a subspace of L^∞, $\overline{C_c(\mathbb{R}^n)} = C_0(\mathbb{R}^n)$.

Theorem 11.2. *Let $1 \leq p, q, \leq \infty$ be conjugate exponents. Let $f \in L^p$ and $g \in L^q$. Then*

1. *$f * g(x)$ exists for each $x \in \mathbb{R}^n$, and*

$$|f * g| \leq \|f\|_p \|g\|_q.$$

2. *$f * g$ is uniformly continuous.*

3. *If $1 < p < \infty$, then $f * g \in C_0(\mathbb{R}^n)$.*

Proof. For $p \neq \infty$, by Hölder's inequality,

$$|f * g(x)| = \left| \int_{\mathbb{R}^n} f(x-y) g(y)\, dy \right| \leq \left(\int_{\mathbb{R}^n} |f(x-y)|^p \, dy \right)^{1/p} \|g\|_q = \|f\|_p \|g\|_q.$$

If $p = \infty$, the proof is easier.

To prove the second statement, it suffices to show that $\lim_{y \to 0} \|\tau_y (f * g) - f * g\|_u = 0$. $\square$

We will finish the proof next time.

12 More Properties of Convolutions and Generalized Young's Inequality

12.1 Uniform continuity and vanishing of convolutions

Let's continue the proof of this statement from last time.

Theorem 12.1. *Let $1 \le p, q, \le \infty$ be conjugate exponents. Let $f \in L^p$ and $g \in L^q$. Then*

1. *$f * g(x)$ exists for each $x \in \mathbb{R}^n$, and*

$$|f * g| \le \|f\|_p \|g\|_q.$$

2. *$f * g$ is uniformly continuous.*

3. *If $1 < p < \infty$, then $f * g \in C_0(\mathbb{R}^n)$.*

Proof. We have already proven the first statement. To prove the second, it suffices to show that

$$\lim_{y \to 0} \|(f * g) - f * g\|_u = 0.$$

Note that if $1 \le p < \infty$,

$$\tau_y(f * g) - f * g = ((\tau_y f) - f) * g.$$

So

$$\|\tau_y(f * g) - f * g\|_u \le \|\tau_y f - f\|_p \|g\|_q \xrightarrow{y \to 0} 0,$$

When $p = \infty$, $q = 1$, and we interchange the role of f and g.

Assume $1 < p < \infty$ so that $1 < q < \infty$. Choose $(f_k)_k, (g_k)_k \in C_c(\mathbb{R}^n)$ such that

$$\lim_{k \to \infty} \|f - f_k\|_p = 0 = \lim_{k \to \infty} \|g - g_k\|_q.$$

By the first proposition stated last time, $f_k * g_k \in C_c(\mathbb{R}^n)$. We have

$$f * g - f_k * g_k = f * (g - g_k) + (f - f_k) * g_k,$$

so

$$\|f * g - f_k * g_k\|_u \le \|f\|_p \|f - f_k\|_q + \|f - f_k\|_p \|g_k\|_q \xrightarrow{k \to \infty} 0.$$

Since $C_0(\mathbb{R}^n)$ is the closure of $C_c(\mathbb{R}^n)$ in the uniform norm, we get the result. $\square$

12.2 Generalized Young's inequality

Theorem 12.2. *Let $1 \leq p, q, r \leq \infty$ be such that $1 + r^{-1} = p^{-1} + q^{-1}$. Let $f \in L^p$.*

1. *(Generalized Young's inequality) If $g \in L^q$, then*
$$\|f * g\|_r \leq \|f\|_p \|g\|_q.$$

2. *Further assume $1 < p, q, r < \infty$ and $g \in$ weak L^q, Then there is a constant $C_{p,q}$ independent of f, g such that*
$$\|f * g\|_r \leq C_{p,q} \|f\|_p [g]_q.$$

3. *If $p = 1$ (so $q = r < \infty$), there exists a constant C_q independent of f such that for any $g \in$ weak L^q,*
$$[f * g]_r \leq C_q \|f\|_1 [g]_q.$$

Proof. For now, we only prove the first statement. Split into cases:

1. $r = \infty$: This is part 1 of the previous theorem (Young's inequality).

2. $p = 1, q = r$: We have already proven this.

3. $1 < p, q, r < \infty$. Since $r^{-1} = p^{-1} + q^{-1} - 1 < q^{-1}$, $q/r \in (0, 1)$. Set $t = 1 - q/r$. Define the operator T as
$$(Tf)(x) = f * g(x) = \int_{\mathbb{R}^n} K(x, y) f(y)\, dy, \qquad K(x, y) = g(x - y).$$
We want to use the Riesz-Thorin interpolation theorem. By Young's inequality,
$$\|T\varphi\|_\infty \leq \|\varphi\|_{\frac{q}{q-1}} \|g\|_q.$$
Also,
$$\|T\varphi\|_q \leq \|\varphi\|_1 \|g\|_q.$$
If we set $p_0 = 1$ and $q_0 = q$ and set $p_1 = q/(q-1)$ and $q_1 = \infty$, then we get that T is weak type $(p_0, q_0$ and (p_1, q_1). Set $t = 1 - q/r \in (0, 1)$, and define $p_t = \frac{1-t}{p_0} + \frac{t}{p_1}$, $q_t = \frac{1-t}{q_0} + \frac{t}{q_t}$. By the Riesz-Thorin theorem,
$$\|Tf\| \leq M_0^{1-t} M_1^t \|f\|_{p_t} = \|g\|_q \|f\|_{p_t},$$
where $M_0 = M_1 = \|g\|_q$. Note that $\frac{1}{q_t} = \frac{1-t}{q} + \frac{t}{\infty} = \frac{q}{r}\frac{1}{q} = \frac{1}{r}$. Similarly, $\frac{1}{p_t} = \frac{1}{p}$.

∎

13 Derivatives of Convolutions

13.1 L^p and weak L^p convolution inequalities

Last time, we proved the first part of the following theorem:

Theorem 13.1. *Let $1 \leq p, q, r \leq \infty$ be such that $1 + r^{-1} = p^{-1} + q^{-1}$. Let $f \in L^p$.*

1. *(Generalized Young's inequality) If $g \in L^q$, then*
$$\|f * g\|_r \leq \|f\|_p \|g\|_q.$$
2. *Further assume $1 < p, q, r < \infty$ and $g \in$ weak L^q, Then there is a constant $C_{p,q}$ independent of f, g such that*
$$\|f * g\|_r \leq C_{p,q} \|f\|_p [g]_q.$$
3. *If $p = 1$ (so $q = r < \infty$), there exists a constant C_q independent of f such that for any $g \in$ weak L^q,*
$$[f * g]_r \leq C_p \|f\|_1 [g]_q.$$

Proof. To complete the proof of the theorem, observe that $[K(x, \cdot)]_q = [g(x - \cdot)]_q = [g]_q < \infty$. Similarly, $[K(\cdot, y)]_q = [g]_q < \infty$. By our interpolation theorem for kernel operators with $c = [g]_q$, we have
$$\|Tf\|_r \leq c\|f\|_p, \qquad p > 1,$$
$$[Tf]_r \leq CB\|f\|_1, \qquad p = 1, r = q.$$
□

13.2 Convolution of C^k functions

Proposition 13.1. *Let $f \in C^k$ be such that $\partial^\alpha f$ is bounded for any $|\alpha| \leq k$, and let $g \in L^1$. Then $f * g \in C^k$ and $\partial^\alpha (f * g) = \partial^\alpha f * g$*

Proof. Proceed by induction on $|\alpha|$. Assume then that $|\alpha| = 1$. Note that
$$f(x + h) = f(x) + \nabla f(x) \cdot h + \int_0^1 (\nabla f(x + th) - \nabla f(x)) \cdot d \, dt.$$

Hence,
$$\begin{aligned} f * g(x + h) &= \int_{\mathbb{R}^n} f(x + h - y) g(y) \, dy \\ &= \int_{\mathbb{R}^n} f(x - y) g(y) \, dy + h \cdot \int_{\mathbb{R}^n} \nabla f(x - y) g(y) \, dy + h \cdot \int_{\mathbb{R}^n} \int_0^1 \varepsilon(t, y) g(y) \, dt \, dy, \end{aligned}$$

where $\varepsilon(t,y) = \nabla f(x-y+th) - \nabla f(x-y)$. Note that $\|\varepsilon\|_u \leq 2\|\nabla f\|_u$. Thus,

$$|\varepsilon(t,y)g(y)| \leq |g(y)|, \qquad g \in L^1((0,1)\times\mathbb{R}^n).$$

So

$$\lim_{h\to 0}\int_{\mathbb{R}^n}\int_0^1 \varepsilon(t,y)g(y)\,dt\,dy = \int_{\mathbb{R}^n}\int_0^1 \lim_{h\to 0}\varepsilon(t,y)g(y)\,dt\,dy = 0.$$

In other words,

$$f*g(x+h) = g*g(x) + h\cdot(\nabla f * g)(x) + h\cdot\gamma(x,h), \qquad \lim_{h\to 0}|\gamma(x,h)| = 0.$$

This proves that $\nabla(f*g)$s exists and equals $\nabla f * g$. □

13.3 Convolution of functions in the Schwarz space

Proposition 13.2. *If $f, g \in \mathcal{S}$, then $f*g \in \mathcal{S}$.*

Proof. By the previous proposition, $f, g \in C^\infty$. Recall that $\|f\|_{(N,\alpha)} = \|(1+|x|)^N \partial\alpha f\|_u$ and that these are bounded for all α, N. Note that

$$(1+|x|) \leq 1 + |x-y| + |y| \leq (1+|x-y|)(1+|y|),$$

and so

$$\begin{aligned}
(1+|x|)^N|\partial^\alpha(f*g(x))| &= (1+|x|)^N|(\partial^\alpha f)*g(x)| \\
&\leq \int_{\mathbb{R}^n}(1+|x-y|)^N|\partial^\alpha f(x-y)|(1+|y|)^N|g(y)|\,dy \\
&= \int_{\mathbb{R}^n}(1+|x-y|)^N|\partial^\alpha f(x-y)|(1+|y|)^{N+n+1}|g(y)|\frac{1}{(1-|y|)^{n+1}}\,dy \\
&\leq \int_{\mathbb{R}^n}\|f\|_{(N,\alpha)}\|g\|_{(N+n+1,0)}\frac{1}{(1+|y|)^{n+1}}\,dy \\
&= |S^{n-1}|\|f\|_{(N,\alpha)}\|g\|_{(N+n+1,0)}\int_0^\infty \frac{r^{n-1}}{(1+r)^{n+1}}\,dr \\
&< \infty.
\end{aligned}$$

□

Remark 13.1. If μ be a measure and $\int_{\mathbb{R}^n} f(x-y)\,d\mu(y)$ makes sense, we denote it as $f*\mu$.

Example 13.1. Let $\mu = \delta_a$. Then

$$\int_{\mathbb{R}^n}\varphi(y)\,d\mu(y) = \varphi(a),$$

and so

$$f * \delta_a(x) = f(x-a).$$

Let $g \in L^1$, and set

$$g_t(x) = \frac{1}{t^n} g(x/t).$$

Then

$$\int_{\mathbb{R}^n} g_t(x)\, dx = \int_{\mathbb{R}^n} g(x/t) d(x/t) = a = \int_{\mathbb{R}^n} g(y)\, dy.$$

We get

$$\int_{\mathbb{R}^n} \varphi(y)\, g_t(y)\ , dy = \int_{\mathbb{R}^n} \varphi(y) g(y/t)\, d(y/t) = \int_{\mathbb{R}^n} \varphi(tz) g(z)\, dz \xrightarrow{t\to 0} \varphi(0) \int_{\mathbb{R}^n} g(z)\, dz = \varphi(0)a.$$

So $g_t \to a\delta_0$.

14 Limits of Scaled Convolutions

14.1 Limits of scaled convolutions

Recall that if $\phi \in L^1(\mathbb{R}^n)$ and
$$\phi_t(x) = \frac{1}{t^n}\phi\left(\frac{x}{t}\right),$$
then
$$\|\phi_t\|_1 = \|\phi\|.$$

Theorem 14.1. *Let $\phi \in L^1$, let $f \in L^p$ and let $1 \le p \le \infty$.*

1. *If $p < \infty$,*
$$\lim_{t\to 0} \|\phi_t * f - af\|_p = 0, \qquad a = \int_{\mathbb{R}^n} \phi(y)\,dy.$$

2. *If $p = \infty$ and f is uniformly continuous, then*
$$\lim_{t\to 0} \|\phi_t * f - af\|_u = 0, \qquad a = \int_{\mathbb{R}^n} \phi(y)\,dy.$$

3. *If $O \subseteq \mathbb{R}^n$ is a bounded open set, $K \subseteq O$ is compact, and $f \in C(O) \cap L^\infty$, then*
$$\lim_{t\to 0} \|\phi_t * f - f\|_{C(K)} = 0.$$

Proof. (1) Assume $1 < p < \infty$, and set $q = p/(p-1)$. We have
$$\phi_t * f(x) - af(x) = \int_{\mathbb{R}^n} (f(x-y) - f(x))\phi_t(y)\,dy = \frac{1}{t^n}\int_{\mathbb{R}^n} (f(x-y) - f(x))\phi\left(\frac{y}{t}\right)\,dy$$
Making the change of variables, $z = y/t$, we get
$$\phi_t * f(x) - af(x) = \int_{\mathbb{R}^N} (f(x-tz) - f(x))\phi(z)\,dz.$$
So
$$\int_{\mathbb{R}^n} |\phi_t * f(x) - af(x)|^p\,dx = \int_{\mathbb{R}^n} \left|\int_{\mathbb{R}^n} (f(x-tz) - f(x))\phi(z)\,dz\right|^p\,dx.$$
Using Minkowski's inequality for integrals, we obtain
$$\begin{aligned}\left(\int_{\mathbb{R}^n} |\phi_t * f(x) - af(x)|^p\,dx\right)^{1/p} &\le \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} |f(x-tz) - f(x)|^p|\phi(z)|^p\,dx\right)^{1/p}\,dz \\ &\le \int_{\mathbb{R}^n} |\phi(z)|\|\tau_{tz}f - f\|_p\,dz\end{aligned}$$

Note that

$$\|\tau_{tz}f - f\|_p \le 2\|f\|_p, \qquad \lim_{t\to 0}\|\tau_{tz}f - f\|_p = 0.$$

So $\|\phi_t * f - f\|_p \le \int_{\mathbb{R}^n} \psi(t,z)\,dz$, where $|\psi| \le 2\|f\|_p|\phi| \in L^1$. Using the dominated convergence theorem, this completes the proof of the first claim. Note that the proof also works for $p = 1$.

(2) Assume $p = \infty$, and let f be uniformly continuous. Set

$$m_f(\delta) = \sup_{|x-y|\le\delta} |f(x) - f(y)|,$$

so that

$$\lim_{\delta\to 0} m_f(\delta) = 0.$$

As we have calculated above,

$$|\phi_t * f(x) - af(x)| \le \int_{\mathbb{R}^n} m_f(t|z|)|\phi(z)|\,dz.$$

But $m_f(t|z|)|phi(z)| \le 2\|f\||\phi| \in L^1$. We apply the dominated convergence theorem to obtain that

$$\limsup_{t\to 0} \phi_t * f - f\|_u \le \limsup_{t\to 0}\int_{\mathbb{R}^n} m_f(t|z|)|\phi(z)|\,dz = 0.$$

So we get the second claim.

(3) Let $2d = \operatorname{dist}(K, O^c)$. Choose a compact $K_1 \subseteq O$ such that $K \subseteq K_1$ and $\operatorname{dist}(K_1, O^c) \ge d$. Fix $\varepsilon > 0$. It suffices to show that

$$\limsup_{t\to 0}\|\phi_t * f - f\|_{C(K)} \le \varepsilon.$$

Let $R > 0$ be large so that

$$\int_{\mathbb{R}^n\setminus B_R(0)} |\phi|\,dz < \frac{\varepsilon}{2(1+\|f\|_\infty)}.$$

Fix $x \in K$. by our earlier calculation,

$$\phi_t * f(x) - af(x) = \underbrace{\int_{B_R(0)} (f(x-tz) - f(x))\phi(z)\,dz}_{I_1(t)} + \underbrace{\int_{\mathbb{R}^n\setminus B_R(0)} (f(x-tz) - f(x)\phi(z)\,dz}_{I_2(t)}.$$

We have

$$|I_2| \le 2\|f\|_\infty \int_{X\setminus B_R(0)} |\phi(z)|\,dz \le \varepsilon.$$

Since K is compact and $f \in C(K_1)$, f is uniformly continuous on K_1, and

$$\lim_{\delta \to 0} m_{K_1}(\delta) = 0, \qquad \text{where } m_{K_1}(\delta) = \sup_{\substack{|z-y| \le \delta \\ z,y \in K_1}} |f(y) - f(z)|.$$

Since $x \in K$ if $tR < d$, then $x, x - tz \in K_1$ if $|z| < R$. So

$$|I_1| \le \int_{B_R(0)} m_{K_1}(tR) |\phi(z)| \, dz = m_{K_1}(tR) \int_{B_R(0)} |\phi(z)|.$$

Hence, $\lim_{t \to 0} I_1(t) = 0$. So we get

$$\limsup_{t \to 0} \|\phi_t * f - f\|_{C(K)} \le \varepsilon. \qquad \square$$

Remark 14.1. Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a Borel function. Assume $c, \varepsilon > 0$ and

$$|\phi(z)| \le \frac{C}{(1 + |z|)^{n+\varepsilon}}$$

for all z. Note that $\phi \in L^p$ for any $p \in [1, \infty]$. Indeed, if $1 \le p < \infty$,

$$\|\phi_p\|^p \le c^p \int_{\mathbb{R}^n} \frac{1}{(1 + |z|)^{(n+\varepsilon)p}} \, dz = C^p |S^{n-1}| \int_0^\infty \frac{r^{n-1}}{(1 + r)^{(n+\varepsilon)p}} \, dr$$

There for, for any $q \in [1, \infty]$ and any $f \in L^q$, $\phi_t * f(x)$ exists.

Our goal is to show thati f x is a Lebesgue point for f, then

$$\lim_{t \to 0} \phi_t * f(x) = f(x).$$

15 Approximation of L^p Functions by Convolutions with Scaled Mollifiers

Today's lecture was given by a guest lecturer.

15.1 Approximation of L^p functions by convolutions with scaled mollifiers

Theorem 15.1. *Suppose $|\phi(x)| \le C(1+|x|)^{-n-\varepsilon}$ for some $C, \varepsilon > 0$ (so $\phi \in L^1(\mathbb{R}^d)$), and let $\int_{\mathbb{R}^d} \phi(x)\,dx = a$. If $f \in L^p$ with $1 \le p \le \infty$, then $f * \phi_t(x) \to af(x)$ as $t \to 0^+$ for every x in the Lebesgue set of f.*

Remark 15.1. This implies that $f * \phi_t(x) \to af(x)$ for a.e. x and for every x for which f is continuous

Proof. If x is in the Lebesgue set of f, for any $\delta > 0$, there exists an $\eta > 0$ such that

$$\int_{B_r} |f(x-y) - f(x)|\,dy \le \delta r^n, \qquad \forall r \le \eta.$$

In other words, $\lim_{r\to 0^+} \frac{1}{|B_r|}\int_{B_r} |f(x-y)-f(x)|\,dy = 0$. We have

$$\begin{aligned}
|f * \phi_t(x) - af(x)| &= \left| \int_{\mathbb{R}^d} f(x-y)\phi_t(y) - f(x)\phi_t(y)\,dy \right| \\
&= \int_{\mathbb{R}^d} |\phi_t(y)||f(x-y)-f(x)|\,dy \\
&= \underbrace{\int_{B_y} |\phi_t(y)||f(x-y)-f(x)|\,dy}_{I_1} + \underbrace{\int_{B_y} |\phi_t(y)||f(x-y)-f(x)|\,dy}_{I_2}.
\end{aligned}$$

We claim that $I_a \le A\delta$ for some A independent of t and that $I_2 \to 0$ as $t \to 0^+$. If the claim holds, then

$$\lim_{t\to 0^+} |f * \phi_t(x) - af(x)| \le \lim_{t\to 0^+} I_1 \le A\delta$$

Letting $\delta \to 0$,

$$\lim_{f\to 0^+} f * \phi_t(x) = af(x).$$

To estimate I_1, let $K \in \mathbb{Z}$ be such that $2^K \le \eta/t \le 2^{K+1}$ if $\eta/t \ge 1$ and $K = 0$ if $\eta/t < 1$. We view the ball B_y as the union of $B_{2^{1-k}\eta} \setminus B_{2^{-k}\eta}$ for $k = 1, 2, 3, \ldots, K$ and the ball $B_{2^{-K}\eta}$. We have a few cases:

1. On $B_{2^{1-k}\eta} \setminus B_{2^{-k}\eta}$ for $k = 1, \ldots, K$,

$$|\phi_t(y)| = t^{-n}\phi(t^{-1}y)| \le Ct^{-n}(1+|t^{-1}y|)^{-n-\varepsilon} \le Ct^{-n}(1+|t^{-1}2^{-k}\eta|)^{-n-\varepsilon}.$$

2. On $B_{2^{-K}\eta}$,

$$|\phi_t(y)| = t^{-n}|\phi(t^{-1}y)| \le Ct^{-n}.$$

So

$$\begin{aligned}
I_1 &= \int_{B_\eta} |\phi_t(y)||f(x-y)-f(x)|\,dy \\
&= \sum_{k=1}^{K} \int_{B_{2^{1-k}\eta}\setminus B_{2^{-k}\eta}} |\phi_t(y)||f(x-y)-f(x)|\,dy + \int_{B_{2^{-K}\eta}} |\phi_t(y)||f(x-y)-f(x)|\,dy \\
&\le \sum_{k=1}^{K} \left(\int_{B_{2^{1-k}\eta}} |f(x-y)-f(x)|\,dy\right) Ct^{-n}|e^{-1}2^{-k}\eta|^{-n-\varepsilon} \\
&\qquad + \left(\int_{B_2^{-K}\eta} |f(x-y)-f(x)|\,dy\right) Ct^{-n} \\
&\le \left(\sum_{k=1}^{K} Ct^{-n}|t^{-1}2^{-k}\eta|^{-n-\varepsilon}\delta(2^{1-k}\eta)^n\right) + Ct^{-n}\delta(2^{-K}\eta)^n \\
&= C\delta\left(\frac{t}{\eta}\right)^{\varepsilon} 2^n \sum_{k=1}^{K} 2^{k\varepsilon} + C_\delta\left(\frac{2^{-K}\eta}{t}\right)^n \\
&= C\delta 2^n \left(\frac{t}{\eta}\right)^{\varepsilon} \frac{2^{(K+1)\varepsilon}-2^{\varepsilon}}{2^{\varepsilon}-1} + C_\delta\left(\frac{2^{-K}\eta}{t}\right)^n
\end{aligned}$$

Use the inequality defining K:

$$\begin{aligned}
&\le C\delta 2^n 2^{-K\varepsilon}\frac{2^{(K+1)\varepsilon}-2^{\varepsilon}}{2^{\varepsilon}-1} + C\delta 2^n \\
&= \underbrace{2^n C(2^{\varepsilon}(2^{\varepsilon}-1)+1)}_{:=A}\,\delta.
\end{aligned}$$

To estimate I_2, we have, using Hölder's inequality,

$$I_n \le \int_{B_\eta^c} (|f(x-y)|+|f(x)|)\,|\phi_t|(y) \le \|f\|_{p'}\|\|\mathbb{1}_{B_\eta^c}\phi_t\|_p + |f(x)|\|\mathbb{1}_{B_\eta^c}\phi_t\|.$$

We split into cases:

1. $p' = \infty$: Then

$$\|\mathbb{1}_{B_\eta^c}\phi_t\|_{p'} \le Ct^{-n}(1+t^{-1}\eta)^{-n-\varepsilon} = Ct^{\varepsilon}(t+\eta)^{-n-\varepsilon} \le Ct^{\varepsilon}\eta^{-n-\varepsilon}.$$

2. $1 \leq p' < \infty$:

$$\begin{aligned}
\|\mathbb{1}_{B_\eta^c}\phi_t\|_{p'} & \\
&= \int_{B_\eta} t^{-np'}|\phi(t^{-1}y))|^{p'}\, dy \\
&= t^{n(1-p')} \int_{B_{\eta/t}^c} |\phi(z)|^p\, dz \\
&\leq C t^{n(1-p')} \int_{B_{\eta/t}^c} \left[(1+|z|)^{-n-\varepsilon}\right]^{p'}\, dz \\
&= C t^{n(1-p')} \left(\frac{\eta}{t}\right)^{n-(n-\varepsilon)p'} \\
&\leq C t^{\varepsilon p'},
\end{aligned}$$

which goes to 0 as $t \to 0^+$. $\square$

Suppose we want to show that C_c^∞ is dense in L^p. Then we let $f_n = f\mathbb{1}_{B_n}$, so $f_n \to f$ in L^p. The idea is then that $f_n * \phi_t \to f - n$ as $t \to 0^+$, so $f_n * \phi_t \in C_c^\infty$ approximates f in L^p.

16 Smooth Density Results, Smooth Urysohn's Lemma, and Characters of $\mathbb{R}^n$

16.1 Density results for C_c^∞ and $\mathcal{S}$

Let $\phi_1 \in C_c^\infty(\overline{B_1(0)})$ be such that $\phi_1 > 0$ on $B_1(0)$ and such that

$$\int_{\mathbb{R}^n} \phi_1(x)\, dx = 1.$$

For example, take

$$\phi(x) := \begin{cases} e^{1/(\|x\|^2-1)} & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}, \qquad \phi_1(x) = \frac{\phi(x)}{\int \phi}.$$

Lemma 16.1. *If $1 \leq p < \infty$, then C_c^∞ and $\mathcal{S}$ are dense in L^p.*

Proof. Let $f \in L^p$, and let $\varepsilon_0 > 0$. We are to find $g \in C_c^\infty$ such tht $\|f-g\|_p < \varepsilon_0$. Choose $\tilde{g} \in C_c$ such that $\|\tilde{g} - f\|_p < \varepsilon_0/2$. Set

$$\phi_\varepsilon(x) = \frac{1}{\varepsilon^n}\phi_1\left(\frac{x}{\varepsilon}\right).$$

We have $\phi_\varepsilon * \tilde{g} \in C^\infty$. Furthermore, $\operatorname{supp}(\phi_\varepsilon * \tilde{g}) \subseteq \operatorname{supp}(\tilde{g}) + \overline{B_\varepsilon(0)}$. Hence, $\phi_\varepsilon * \tilde{g} \in C_c^\infty$. Choose ε small enough such that

$$\|\phi_\varepsilon * \tilde{g} - \tilde{g}\|_p < \varepsilon/2.$$

So the desired inequality holds for $g = \phi_\varepsilon * \tilde{g}$. In conclusion, $L^p \subseteq \overline{C_c^\infty}^{L^p} \subseteq \overline{\mathcal{S}}^{L^p} \subseteq L^p$. $\square$

Lemma 16.2. *C_c^∞ and $\mathcal{S}$ are dense in $C_0(\mathbb{R}^n)$ for the uniform norm.*

Proof. Let $f \in C_0$. Recall that $\overline{C_c}^{L^\infty} = C_0$. Hence, given $\varepsilon_0 > 0$, there is a $\tilde{g}\ inC_c$ such that $\|f - \tilde{g}\|_u \leq \varepsilon/2$. Since $\tilde{g}$ is uniformly continuous and bounded,

$$\lim_{\varepsilon \to 0} \|\tilde{g} - \phi_\varepsilon * \tilde{g}\|_u = 0.$$

Thus , there is $\varepsilon > 0$ such that

$$\|\tilde{g} - \phi_\varepsilon * \tilde{g}\|_u \leq \varepsilon/2.$$

So we get

$$\|f - g\|_u \leq \varepsilon.$$

If we set $g = \phi_\varepsilon * \tilde{g}$, as before, $g \in C_c^\infty$. In conclusion, $C_0 \subseteq \overline{C_c^\infty}^{L^\infty} \subseteq \overline{\mathcal{S}}^{L^\infty} \subseteq C_0$. $\square$

16.2 Smooth Urysohn's lemma

Lemma 16.3 (Urysohn)**.** *Let $K \subseteq \mathbb{R}^n$ be a compact nonempty set, and let $U \subseteq \mathbb{R}^n$ be an open set such that $K \subseteq U$. Then there exists a function $f \in C_c^\infty$ such that $f|_K = 1$,* $\mathrm{supp}(f) \subseteq U$*, and* $\mathrm{supp}(f)$ *is compact.*

This is useful on manifolds. Treat a neighborhood of a point as a subset of $\mathbb{R}^n$. If you want to integrate a function on the manifold, you can integrate it over every neighborhood.

Proof. Set $3\delta = \mathrm{dist}(K, U^c) > 0$. Let $K_1 = \{x \in \mathbb{R}^n : \mathrm{dist}(x, K) \leq \delta\}$. Note that K_1 is compact, and $\mathrm{dist}(K_1, U^c) \geq \delta$. Let $f = \phi_\varepsilon * \mathbb{1}K_1$. Then $f \in C^\infty$, and $\mathrm{supp}(f) \subseteq K_1 + \overline{B_\varepsilon(0)} \subseteq U$ if $\varepsilon < \delta$. So $f \in C_c^\infty$, and f has compact support. If $x \in K$, then

$$f(x) = \int_{\mathbb{R}^n} \mathbb{1}_{K_1}(x-y)\phi_\varepsilon(y)\,dy = \int_{\overline{B_\varepsilon(0)}} \mathbb{1}_{K_1}(x-y)\phi_\varepsilon(y)\,dy.$$

If $x \in K$ and $|y| \leq \varepsilon$, then $x - y \in K_1$. So

$$f(x) = \int_{\overline{B_\varepsilon(0)}} \phi_\varepsilon(y)\,dy = \int_{\mathbb{R}^n} \phi_\varepsilon(y)\,dy = 1. \qquad \square$$

16.3 Characters on $(\mathbb{R}^n, +)$

Proposition 16.1. *Let $\phi : \mathbb{R}^n \to \mathbb{C}$ be a measurable function such that $|\phi(x)| = 1$ and $\phi(x+y) = \phi(x)\phi(y)$ for any $x, y \in \mathbb{R}^n$. Then*

1. *There exists $\xi \in \mathbb{R}^n$ such that $\phi(x) = e^{2\pi i \xi \cdot x}$.*
2. *If we further assume that ϕ is $\mathbb{Z}^n$-periodic, then $\xi \in \mathbb{Z}^n$.*

Proof. Let $(e_j)_{j=1}^n$ the standard orthonormal basis of $\mathbb{R}^n$. If $x \in \mathbb{R}^n$, then

$$\phi(x) = \phi\left(\sum_{j=1}^n x_j e_j\right) = \prod_{i=1}^n \phi(x_j e_j).$$

The function $x \mapsto \phi(x, e_j)$ can be identified with the restriction of ϕ to a 1-dimensional space. Therefore, $t \mapsto \phi(te_j)$ satisfies the assumption of the lemma for $n = 1$. It is not a loss of generality to assume $n = 1$.

Set

$$F(x) = \int_0^x \phi(t)\,dt.$$

There exists $a \neq 0$ such that $F(a) \neq 0$ (lest $\phi \equiv 0$). Set $A = 1/F(a)$. We have

$$\phi(a) = \phi(x) A \int_0^a \phi(t)\,dt = A \int_0^a \phi(x)\phi(t)\,dt = A \int_0^a \phi(x+t)\,dt = A \int_x^{x+a} \phi(z)\,dz$$

$$= A(F(x+a) - F(x)).$$

Since F is continuous, this gives us that ϕ is continuous. Since ϕ is continuous, F is continuously differentiable. Apply the equality above again to conclude that $\phi \in C^1$. Differentiating both sides, we obtain

$$\phi'(x) = A(\phi(x+a) - \phi(x)) = A(\phi(x)\phi(a) - \phi(x)) = A\phi(x)(\phi(a) - 1) = \phi(x)B.$$

So we get

$$\frac{d}{dx}\ln(\phi(x)) = \frac{\phi'(x)}{\phi(x)} = B.$$

Integrating, we obtain

$$\phi(x) = \phi(0)e^{Bx}.$$

But $\phi(0) = \phi(0+0) = \phi(0)^2$, and so $\phi(0) \in \{0, 1\}$. Since $|\phi(0)| = 1$, we conclude that $\phi(0) = 1$. Write $B = B_1 + iB_2$ with$B_1, B_2 \in \mathbb{R}$. Then

$$\phi(x) = e^{B_1 x}e^{iB_2 x},$$

and

$$1 = |\phi(1)| = e^{B_1} \implies B_1 = 0.$$

So $\phi(x) = e^{2\pi i \xi x}$, where$\xi = B_2/(2\pi)$.

Assume ϕ is $\mathbb{Z}$-periodic. Then

$$e^{2\pi i \xi} = \phi(1) = \phi(0) = 1.$$

So $2\pi i\xi \in 2\pi\mathbb{Z}$. That is, $\xi \in 2\pi\mathbb{Z}$. □

17 Orthonormal Basis of L^2 and the Fourier Transform

17.1 An orthonormal basis of $L^2(\mathbb{T})$

If $\xi \in \mathbb{R}^n$, we define $E_\xi : \mathbb{R}^n \to \mathbb{C}$ by $E_\xi(x) = e^{2\pi i x\cdot\xi}$, where $x \cdot \xi = \sum_{i=1}^n x_i \xi_i$. Let $\mathcal{E} = \{E_k : k \in \mathbb{Z}^n\}$.

Proposition 17.1. *$\mathcal{E}$ separates points in $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$: For $a, b \in \mathbb{T}^n$, if $E_k(a) = E_k(b)$ for all $k \in \mathbb{Z}^n$, then $a = b$.*

Proof. Assume $E_k(a) = E_k(b)$. Then $e^{2\pi i a\cdot k} = e^{2\pi i b\cdot k}$, so $e^{2\pi i(b-a)\cdot k} = 1$. So $\cos(2\pi(b - a)\cdot k) = 1$, and $\sin(2\pi(b-a)\cdot k) = 0$. This means $(b-a)\cdot k \in \mathbb{Z}$, and this holds for all $k \in \mathbb{Z}^k$. In particular, taking $k = (0, \dots, 0, 1, 0, \dots, 0)$, we conclude that $e^{2\pi i(b_j - a_j)} = 0$ for $j = 1, \dots, n$. So $b_j - a_j \in \mathbb{Z}$, which means that $b_j - a_j = 0$ (since $a, b \in [0,1)^n$ and $|a_j - b_j| < 1$). $\square$

Theorem 17.1. *The collection $\mathcal{E}$ is an orthonormal basis of $L^2(\mathbb{T}^n)$ for the inner product $\langle f, g\rangle = \int_{\mathbb{T}^n} f(x)\overline{g(x)}\,dx = \int_{[0,1]^n} f(x)\overline{g(x)}\,dx$.*

Proof. Let $k, \ell, \in \mathbb{Z}^n$. Then

$$\langle E_k, E_\ell\rangle = \int_{[0,1]^n} e^{2\pi i(k-\ell)\cdot x}\,dx = \prod_{j=1}^n \int_0^1 e^{2\pi i(k_j - \ell_j)x_j}\,dx_j = \begin{cases} 1 & k = \ell \\ 0 & k \neq \ell. \end{cases}$$

So $\mathcal{E}$ is orthonormal.

It remains to show that $\mathcal{E}$ spans a dense subset of $L^2(\mathbb{T}^n)$. Let $A = \{\sum_{k\in\Lambda} \lambda_k E_k : \Lambda \subseteq \mathbb{Z}^n \text{ is finite}, \lambda_k \in \mathbb{C}\}$. Since $E_k E_\ell \in \mathcal{E}$ for any $k, \ell \in \mathbb{Z}^n$, one checks that $\mathcal{A}$ is an algebra in $C(\mathbb{T}^n)$. Since $\mathbb{C} = \{\lambda E_0 : \lambda \in \mathbb{C}\}$, we conclude that $\mathcal{A}$ contains the constant functions. By the Stone-Weierstrass theorem, $\mathcal{A}$ is dense in $C(\mathbb{T}^n)$ for the uniform norm. If $f \in L^2(\mathbb{T}^n)$ and $\varepsilon > 0$, there exists $g \in C(\mathbb{T}^n)$ such that $\|f - g\|_2 < \varepsilon/2$. Choose $h \in \mathcal{A}$ such that $\|g - h\|_u \,\mathrm{eq}\, \varepsilon/2$. Then $\|g - h\|_2 \leq \|g - h\|_u$, since $m(\mathbb{T}) = 1$. Consequently, $\|f - h\|_2 < \varepsilon$. This proves that $\mathcal{A}$ is dense in $L^2(\mathbb{T})$. $\square$

17.2 The Fourier transform

Remark 17.1. Let $f \in L^2(\mathbb{T}^n)$. Then

$$f = \sum_{k\in\mathbb{Z}^n} \langle f, E_k\rangle\, E_k, \qquad \|f\|_2^2 = \sum_{k\in\mathbb{Z}^n} |\langle f, E_k\rangle|^2.$$

Set

$$\widehat{f}_k = \langle f, E_k\rangle\,, \qquad \widehat{f} = (\widehat{f}_k)_{k\in\mathbb{Z}^n}.$$

We have a map $\Lambda : L^2(\mathbb{T}^n) \to \ell^2(\mathbb{Z}^n)$ sending $f \mapsto \widehat{f}$. This is an isometry because this relation gives $\|f\|_2 = \|\widehat{f}\|_2$ (Parseval's identity).

Remark 17.2. Observe that if $f \in L^1(\mathbb{T}^n)$, since $E_k \in L^\infty(\mathbb{T}^n)$, we have $fE_k \in L^1(\mathbb{T}^n)$, and so $\widehat{f}_k$ is still well defined. Note that

$$|\widehat{f}_k| = \left| \int_{\mathbb{T}^n} f(x) e^{2\pi i k \cdot x} \, dx \right| \leq \|f\|_1.$$

In other words,

$$\|\widehat{f}\|_{\ell^\infty(\mathbb{Z}^n)} \leq \|f\|_1.$$

Theorem 17.2. *Let $1 < p < 2$, and let $q = p/(p-1)$ be the conjugate exponent. Then the Fourier transform Λ extends to a linear map $\Lambda : L^p(\mathbb{R}^n) \to \ell^q(\mathbb{Z}^n)$ such that*

$$\|\widehat{f}\|_{\ell^q(\mathbb{Z}^n)} \leq \|f\|_{L^p(\mathbb{T}^n)}.$$

Proof. We want to apply the Riesz-Thorin theorem. Set $p_0 = 2$ and $p_1 = 1$, so $q_0 = 2$ and $q_1 = \infty$. Set $t = 2/p - 1 \in (0,1)$, and set

$$\frac{1}{p_t} := \frac{t}{p_0} + \frac{1-t}{p_0} = \frac{1}{p}, \qquad \frac{1}{q_t} := \frac{t}{q_0} + \frac{1-t}{q_0} = \frac{1}{q}.$$

By the Riesz-Thorin interpolation theorem,

$$\|\widehat{f}\|_{\ell^q(\mathbb{Z}^n)} \leq M_1^t M_0^{1-t} \|f\|_{L^p(\mathbb{T}^n)} = \|f\|_{L^p(\mathbb{T}^n)},$$

as $M_0 = M_1 = 1$. $\square$

18 Properties of The Fourier Transform

18.1 Properties of the Fourier transform

If $f \in L^2(\mathbb{T}^n)$ and $k \in \mathbb{Z}^n$, then $\widehat{f}(k) = \langle f, E_k \rangle = \int_{\mathbb{R}}^n f(x) e^{-2\pi i k \cdot x}\, dx$.

Definition 18.1. The **Fourier series** are

$$\sum_{k \in \Lambda} \widehat{f}(k) E_k$$

for $\Lambda \subseteq \mathbb{Z}^n$.

Definition 18.2. Let $f \in L^1(\mathbb{R}^n)$. As $E_\xi \in L^\infty(\mathbb{R}^n)$,, $\widehat{f}(\xi) = \langle f, E_\xi \rangle = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \xi \cdot x}\, dx$. The **Fourier transform** of f at ξ is

$$(\mathcal{F} f)(\xi) = \widehat{f}(\xi).$$

Proposition 18.1. *Let $f, g \in L^1(\mathbb{R}^n)$, let $y, \eta \in \mathbb{R}^n$ and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be an invertible linear map.*

1. $\widehat{f * g} = \widehat{f} \cdot \widehat{g}$.
2. $\widehat{f} \in C_b(\mathbb{R}^n)$.
3. $\widehat{\tau_y f} = \widehat{f}\overline{E_y}$ *and* $\tau_\eta(\widehat{f}) = \widehat{f E_\eta}$.
4. *If* $S = T^{-1}$, *then* $\widehat{f \circ T} = |\det(S)| \widehat{f} \circ S^\top$.
5. *For* $t > 0$, *set* $f_t(x) = t^{-n} f(x/t)$. *Then* $\mathcal{F}(f_t) = (\mathcal{F}(f))_t$.

Proof. 1. Let $\xi \in \mathbb{R}^n$. Then

$$\begin{aligned}
\widehat{f * g}(\xi) &= \int_{\mathbb{R}^n} f * g(x) e^{-2\pi i \xi \cdot x}\, dx \\
&= \int_{\mathbb{R}^n} e^{-2\pi i \xi \cdot x} \int_{\mathbb{R}^n} f(x-y) g(y)\, dy\, dx
\end{aligned}$$

We can use Fubini's theorem because the product of integrable functions in separate variables is integrable.

$$= \int_{\mathbb{R}^n} g(y) e^{-2\pi i \xi \cdot y} \int_{\mathbb{R}^n} f(x-y) e^{-2\pi i \xi \cdot (x-y)}\, dx\, dy$$

Make the change of variables $z = x - y$:

$$\begin{aligned}
&= \int_{\mathbb{R}^n} g(y) e^{-2\pi i \xi \cdot y} \int_{\mathbb{R}^n} f(z) e^{-2\pi i \xi \cdot z}\, dz\, dy \\
&= \int_{\mathbb{R}^n} g(y) e^{2\pi i \xi \cdot y} \widehat{f}(\xi)\, dy \\
&= \widehat{f}(\xi) \widehat{g}(\xi).
\end{aligned}$$

2. We have $|\widehat{f}| \le \|f\|_1$. If $h \in \mathbb{R}^n$,

$$\widehat{f}(\xi + h) = \int_{\mathbb{R}^n} f(x) e^{2\pi i \xi \cdot x} e^{2\pi i h \cdot x}\, dx,$$

so

$$|\widehat{f}(\xi + h) - \widehat{f}(\xi)| \le \int_{\mathbb{R}^n} |f(x)||e^{-2\pi i h\cdot x} - 1|\, dx$$

$|f||e^{-2\pi i \xi \cdot x} - 1| \le 2|f| \in L^2$, so we may apply the dominated convergence theorem to conclude that

$$\limsup_{h\to 0} |\widehat{f}(\xi + h) - \widehat{f}(\xi)| \le \int_{\mathbb{R}^n} \limsup_{h \to 0} |e^{-2\pi i h \cdot x} - 1||f(x)|\, dx = 0.$$

3. Let $\xi \in \mathbb{R}^n$. Then

$$(\widehat{f})(\xi) = \widehat{f}(\xi - \eta) = \int_{\mathbb{R}^n} e^{-2\pi i(\xi - \eta)\cdot x} f(x)\, dx = \int_{\mathbb{R}^n} e^{2\pi i \xi \cdot x} E_k(x) f(x)\, dx = \widehat{E_k f}(\xi).$$

4.

$$\widehat{f \circ T}(\xi) = \int_{\mathbb{R}^n} f \circ T(x) e^{-2\pi i \xi \cdot x}\, dx$$

Make the change of variables $y = Tx$, so $x = Sy$ and $dx = |\det(S)|\, dy$.

$$= \int_{\mathbb{R}^n} f(y) e^{-2\pi i \xi \cdot Sy} |\det(S)|\, dy$$

Use the fact that $a \cdot (Sb) = S^\top a \cdot b$:

$$\begin{aligned} &= \int_{\mathbb{R}^n} f(y) e^{-2\pi i S^\top \xi \cdot y} |\det(S)|\, dy \\ &= |\det(S)| \widehat{f} \circ S^\top(\xi). \end{aligned}$$

5. Set $Tx = x/t$. so $Sy = ty$. Define $p_t(f)(x) = t^{-n} f(x/t) = |\det(S)|^{-1} f \circ T(x)$. By the previous part,

$$\widehat{O_t(f)} = \frac{1}{|\det(S)|} \widehat{f \circ T} = \frac{1}{|\det(S)|} |\det(S)| \widehat{f} \circ S^\top = \widehat{f}(t\xi) = t^n O_{1/t} \circ \widehat{f}. \qquad \square$$

19 The Fourier Transform and Derivatives

19.1 How the Fourier transform interacts with derivatives

Theorem 19.1. *Let $f \in L^1$. Then the following hold.*

1. *If $x^\alpha f \in L^1$ for all $|\alpha| \leq k$, then*
$$\partial^\alpha \widehat{f}(\xi) = \widehat{(-2\pi i x)^\alpha f}(\xi).$$

2. *If $f \in C^k$, $\partial^\alpha f \in L^1 \cap C_0$ for $|\alpha| \leq k-1$, and $\partial^\alpha f \in L^1$ for $|\alpha| = k$, then*
$$\widehat{\partial^\alpha f}(\xi) = (2\pi i \xi)^\alpha \widehat{f}(\xi).$$

Proof. For the first statement, we will show the proof of $|\alpha| = 1$. The rest will follow by induction on $|\alpha|$. Let $\xi \in \mathbb{R}^n$. Then
$$\widehat{f}(\xi + h) = \int_{\mathbb{R}^n} e^{-2\pi i \xi \cdot x} e^{-2\pi i h \cdot x} f(x)\, dx.$$
If $h = te_j$, then
$$\frac{\widehat{f}(\xi + te_j) - \widehat{f}(\xi)}{t} = \int_{\mathbb{R}^n} e^{2\pi i \xi \cdot x} \frac{e^{-2\pi i t x_j} - 1}{t}\, dx.$$
Using a first order Taylor expansion of the exponential, we get $|e^{-2\pi i x_j t}|/|t| \leq 2\pi |x_j|$. So, using the dominated convergence theorem, since $2\pi |x_j||f(x)| \in L^1$,
$$\partial^\alpha \widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{2\pi i \xi \cdot x}(-2\pi i x_j)\, dx = \widehat{(-2\pi i x)f}(\xi).$$

For the second statement, we want to understand why we need $f \in C_0 \cap L^1$ and $\partial_{x_j} f \in L^1$ to have $\widehat{\partial_{x_j} f}(\xi) = (2\pi i \xi_j)\widehat{f}(\xi)$. Assume $k = 1$. Then
$$\begin{aligned}
\widehat{\partial_{x_j} f}(\xi) &= \int_{\mathbb{R}^n} \partial_{x_j} f(x) e^{-2\pi i \xi \cdot x}\, dx \\
&= \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \partial_{x_j} f(x) e^{-2\pi i \sum_{k \neq j} \xi_k x_k} e^{-2\pi i x_j \xi_j}\, dx_j\, dx_1 \cdots dx_{j-1}\, dx_j \cdots dx_n \\
&= \int_{\mathbb{R}^{n-1}} e^{-2\pi i \tilde{\xi \cdot x}^{-j}} \left[-\int_{\mathbb{R}} f(x) \frac{\partial}{\partial_{x_j}} \left(e^{2\pi i \xi_j x_j} \right) + \left[f(x) e^{-2\pi i x_j \xi_j} \right]_{-\infty}^{\infty} \right] \tilde{dx}^{-j} \\
&= \int_{\mathbb{R}^n} e^{2\pi i x \cdot \xi} f(x)(-2\pi i \xi_j)\, dx \\
&- 2\pi i \xi_j \widehat{f}(\xi)
\end{aligned}$$

To prove that $\widehat{f} \in C_0$, it suffices to find $(g_k)_k \subseteq C_0$ such that $\lim_k \|\widehat{f} - g_k\|_u = 0$. Let $(f_k)_k \subseteq C_0^\infty(\mathbb{R}^n)$ be such that $\|f - f_k\|_1 \leq 1/k$. We have

$$\|\widehat{f} - \widehat{f}_j\|_u \leq \|f - f_k\|_1 \leq \frac{1}{k}.$$

But $(2\pi i \xi_j)\widehat{f}_k = \widehat{\partial_{x_j} f_k}$. Thus,

$$2\pi\|\xi_j \widehat{f}_k\|_u \leq \|\partial_{x_j} f_k\|_1 < \infty.$$

This means that $|\xi||\widehat{f}_k|$ is bounded, and so $\widehat{f}_k \in C_0$. □

19.2 The Fourier transform on the Schwarz space

Corollary 19.1. *$\mathcal{F}$ maps $\mathcal{S}$ into $\mathcal{S}$ continuously.*

Proof. Let $f \in \mathcal{S}$. We are to control the uniform norm of $x^a \partial^b \widehat{f}$ for all multi-indices $a, b \in \mathbb{N}^n$ using a finite number of expressions $\|f\|_{(N_i, \alpha_i)}$. Since $x^a \partial^b \widehat{f}$ is a finite linear combination of terms of the form $\partial^\beta(x^\alpha \widehat{f})$, it suffices to control the latter expressions. Note that

$$\partial^\beta(x^\alpha \widehat{f}) = \frac{\partial^\beta \left((2\pi i x)^\alpha \widehat{f} \right)}{(2\pi i)^\alpha} = \frac{\partial^\beta(\widehat{\partial^\alpha f})}{(2\pi i)^\alpha} = \frac{1}{(2\pi i)^\alpha} \widehat{(-2\pi i x)^\beta \partial^\alpha f}.$$

Thus,

$$\|\partial^\beta(x^\alpha \widehat{f})\|_u \leq |2\pi|^{\beta - \alpha} \|x^\beta \partial^\alpha f\|_1.$$

The right hand side is

$$\begin{aligned}
&|2\pi|^{\beta-\alpha} \int_{\mathbb{R}^n} \frac{1}{(1+|x|)^{n+1}} (1+|x|)^{n+1} |x^\beta \partial^\alpha f| \, dx \\
&\quad \leq |2\pi|^{\beta-\alpha} \|(1+|x|)^{n+1+|\beta|} \partial^\alpha f\|_u \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{1}{1+|x|)^{n+1}} \, dx \\
&\quad = |2\pi|^{\beta-\alpha} \|f\|_{(2+1+|\beta|, \alpha)} C_n,
\end{aligned}$$

where C_n is a constant. □

Remark 19.1. Given $a > 0$ and an integer $n \geq 1$, we define

$$f_a^n(x) = e^{-\pi |x|^2 a}.$$

Note that $f_a^n \in \mathcal{S}$, and

$$f_a^n(x) = \prod_{j=1}^n f_a^1(x_j).$$

Hence,

$$\begin{aligned}\widehat{f_a^n}(\xi) &= \int_{\mathbb{R}^n} e^{-2\pi i \xi \cdot x} \prod_{j=1}^{n} f_a^1(x_j)\, dx \\ &= \prod_{j=1}^{n} \int_{\mathbb{R}^n} e^{2\pi i \xi_j x_j} f_a^1(x_j)\, dx_j \\ &= \prod_{j=1}^{n} \widehat{f_a^1}(\xi_j).\end{aligned}$$

20 Fourier Inversion

20.1 Fourier transform of exponentials

For $a > 0$, recall that

$$f_a^n(x) = e^{-\pi a|x|^2} = \prod_{j=1}^n f_a^1(x_j).$$

Additionally,

$$\int_{\mathbb{R}} \frac{e^{-|x_j-u|^2/2\theta}}{\sqrt{2\pi\theta}}\,dx_j = 1 \implies \int_{\mathbb{R}^n} \frac{e^{-|x-u|^2/2\theta}}{\sqrt{2\pi\theta}}\,dx = 1.$$

Lemma 20.1. *We have*

$$\widehat{f_a^n} = \frac{1}{\sqrt{a}^n} f_{1/a}^n.$$

Proof. Note that

$$\begin{aligned}\widehat{f_a^n}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i\xi\cdot x} f_a^n(x)\,dx &= \prod_{j=1}^n \int_{\mathbb{R}} e^{-2\pi i\xi_j x_j} f_a^1(x_j)\,dx_j \\ &= \prod_{j=1}^n \widehat{f_a^1}(\xi_j).\end{aligned}$$

So it suffices to show the lemma for $n = 1$. Assume $n = 1$.

We want to show that

$$e^{(\pi/a)\xi^2}\widehat{f_a^1}(\xi) = 1.$$

We claim that for $f = f_a^1$,

$$\frac{d}{d\xi}\left(\widehat{f}(\xi)e^{(\pi/a)\xi^2}\right) = 0.$$

We have

$$\frac{d}{d\xi}\widehat{f} = \widehat{-2\pi i x f} = \frac{i}{a}\widehat{2\pi x a e^{-\pi a|x|^2}} = \frac{i}{a}\widehat{\frac{d}{dx}(e^{-\pi a|x|^2})} = -\frac{i}{a}\widehat{\frac{df}{dx}} = -\frac{i}{a}(-2\pi i\xi)\widehat{f} = -\frac{2\pi}{a}\xi\widehat{f}(\xi).$$

Hence,

$$\begin{aligned}\frac{d}{d\xi}(\widehat{f}(\xi)e^{(\pi/a)\xi^2}) &= \frac{d}{d\xi}\widehat{f}(\xi)e^{(\pi/a)\xi^2} + \widehat{f}(\xi)\frac{2\pi}{a}\xi e^{-\pi a\xi^2} \\ &= \xi\frac{2\pi}{a}\widehat{f}(\xi)\left[-e^{(\pi/a)\xi^2} + e^{(\pi/a)\xi^2}\right] \\ &= 0.\end{aligned}$$

Consequently,

$$e^{(\pi/a)\xi^2}\widehat{f_a^1}(\xi) = \widehat{f}(0) = \int_{\mathbb{R}} f(x)\,dx = \int_{\mathbb{R}} e^{-\pi x^2 a}\,dx = \frac{1}{\sqrt{a}}.$$ □

20.2 Self-adjoint property of the Fourier transform

Lemma 20.2. *Let $f, g \in L^1$. Then*

$$\int_{\mathbb{R}^n} \widehat{f} g \, d\xi = \int_{\mathbb{R}^n} f \widehat{g} \, dx.$$

Proof. We have

$$\begin{aligned}
\int_{\mathbb{R}^n} f(x)\widehat{g}(x)\,dx &= \int_{\mathbb{R}^n} f(x) \int_{\mathbb{R}^n} e^{-2\pi i \xi\cdot x} g(\xi)\,d\xi\,dx \\
&= \int_{\mathbb{R}^n\times\mathbb{R}^n} f(x)g(\xi)e^{-2\pi i\xi\cdot x}\,dx\,d\xi \\
&= \int_{\mathbb{R}^n\times\mathbb{R}^n} f(\xi)g(x)e^{-2\pi i\xi\cdot x}\,dx\,d\xi \\
&= \int_{\mathbb{R}^n} \widehat{f}(\xi)g(\xi)\,d\xi.
\end{aligned}$$

□

20.3 The Fourier inversion formula

Definition 20.1. Let $F \in L^1$. We define

$$F^\vee = \widehat{F}(-\xi) = \int_{\mathbb{R}^n} e^{2\pi i\xi\cdot x} F(x)\,dx.$$

Theorem 20.1. *Suppose $F, \widehat{F} \in L^1$. There exists $G \in C_0$ such that $F = G$ a.e. and $(F^\vee)^\wedge = (\widehat{F})^\vee = G$.*

Proof. For each $x \in \mathbb{R}^n$ and $t > 0$, define

$$\phi_t^x(\xi) = e^{2\pi i\xi\cdot x - \pi|\xi|^2 t} = E_x(\xi) f_t^n(\xi).$$

Note that

$$\widehat{\phi_t^x}(y) = \langle E_x f_t^n, E_y\rangle = \langle f_t^n, E_{y-x}\rangle = \widehat{f}_t(y-x) = \frac{1}{\sqrt{t}^n} f_{1/t}^n(y-x) = \frac{1}{\sqrt{t}^n} f_{1/t}^n(x-y).$$

by the lemma. But setting $t = 2\pi\theta$ gives

$$\int_{\mathbb{R}^n} f_{1/t}^n(z) \frac{1}{\sqrt{t}^n}\,dz = 1.$$

In conclusion, $\phi_t^x, \widehat{\phi_t^x} \in L^1$. By the previous lemma,

$$\int_{\mathbb{R}^n} \widehat{\phi_t^x} F(\xi)\,d\xi = \int_{\mathbb{R}^n} \phi_t^x(y)\widehat{F}(y)\,d\xi.$$

Using the expression of $\widehat{\phi_t^x}$, we obtain

$$\int_{\mathbb{R}^n} \frac{1}{\sqrt{t}^n} f^n_{1/t}(x-\xi)F(\xi)\,d\xi = \int_{\mathbb{R}^n} \widehat{F}(y)e^{2\pi i y\cdot x - \pi|y|^2 t}\,dy.$$

Hence

$$\lim_{t\to 0} \rho_t * F = \lim_{t\to 0} \int_{\mathbb{R}^n} \widehat{F}(y)e^{2\pi i y\cdot x-\pi|y|^2 t}\,dy, \qquad \text{where } \rho_t(z) = \frac{1}{\sqrt{t}^n} f^n_{1/t}(z).$$

By the dominated convergence theorem,

$$\lim_{t\to 0} \rho_t * F = \int_{\mathbb{R}^n} \widehat{F}(y)e^{2\pi i y\cdot x}\,dx = (\widehat{F})^{\vee}(-x),$$

and $F = \lim_{t\to 0} \rho_t * F$ a.e., as

$$\rho_t(z) = \frac{1}{\sqrt{t}^n} e^{-\pi|z/\sqrt{t}|^2} = \frac{1}{\sqrt{t}^n}\rho_1(z/\sqrt{t}).$$

So we have proven that

$$F(x) = (\widehat{F})^{\wedge}(-x) = (\widehat{F})^{\vee}(x) \qquad a.e.$$

We have shown that $(\widehat{F})^{\wedge} = (F \circ O)$, where $O(z) = -z$. Now $F^{\vee} = \widehat{F} \circ 0$, so $(F^{\vee})^{\wedge} = (\widehat{F} \circ O)^{\wedge}$. $\square$

21 Isomorphism, Unitary Property of the Fourier Transform, and Periodic Functions

21.1 The Fourier transform on the Schwarz space

If $f, \widehat{f} \in L^1$, then $f \overset{\text{a.e.}}{=} (f^\vee)^\wedge$, where, $f^\vee = \widehat{f} \circ O$, and $O(x) = -x$.

Corollary 21.1. *If $f \in L^1$ and $\widehat{f} = 0$, then $f \equiv 0$ a.e.*

Proof. We have $f, \widehat{f} \in L^1$, and so

$$f \equiv (f^\vee)^\wedge = (\widehat{f} \circ 0)^\wedge = 0^\wedge = 0. \qquad \square$$

Corollary 21.2. *$\mathcal{F} : \mathcal{S} \to \mathcal{S}$ is an isomorphism.*

Proof. By the previous corollary, the kernel of $\mathcal{F}|_{\mathcal{S}}$ is $\{0\}$. Since $\mathcal{F}$ is linear, we conclude that $\mathcal{F}|_{\mathcal{S}}$ is one-to-one. We want to show that $\mathcal{F}|_{\mathcal{S}}$ is onto. Let $g \in \mathcal{S}$. Since $\widehat{g} \in \mathscr{S}$, we have $\widehat{g}, g \in \mathcal{S}$, and so $g = \widehat{\widehat{g} \circ O} = \mathcal{F}(\widehat{g} \circ O)$. Since $\widehat{g} \cong O \in \mathcal{S}$, we have proven that $\mathcal{F}^{-1}(g) = \widehat{g} \circ O$. That is, $\mathcal{F}^{-1} = \mathcal{F} \circ O$. Since $\mathcal{F}$ maps $\mathcal{S}$ continuous to $\mathcal{S}$, so does $\mathcal{F} \circ O = \mathcal{F}^{-1}$. $\square$

21.2 Unitary property of the Fourier transform

Theorem 21.1. *The Fourier transform has the following properties:*

1. *$\mathcal{F}$ maps $L^1 \cap L^2$ into L^2.*
2. *$\mathcal{F}$ extends to a unitary transformation $\tilde{\mathcal{F}} : L^2 \to L^2$.*

Proof. Set $A = \{f \in L^1 : \widehat{f} \in L^1\}$. We claim that $A \subseteq L^2$. Let $f \in A$. Then $f = (f^\vee)^\wedge$ a.e. This is in L^∞, as $\widehat{f} \in L^1$. Since $\frac{1}{2} = \frac{1/2}{1} + \frac{1/2}{\infty}$, we conclude that

$$\|f\|_2 \leq \|f\|_\infty^{1/2} \|f\|_1^{1/2}.$$

Observe that $L^2 = \overline{\mathcal{S}}^{L^2} \subseteq \overline{A}^{L^2} \subseteq L^2$. So A is dense in L^2.

Isometry: Let $f, g \in A$. We have

$$\int_{\mathbb{R}^n} f\overline{g} = \int_{\mathbb{R}^n} f(\overline{g}^\vee)^\wedge = \int_{\mathbb{R}^d} \widehat{f}\overline{g}^\vee = \int \widehat{f}\overline{\widehat{g}}.$$

In particular,

$$\int_{\mathbb{R}^n} |f|^2 \, dx = \int_{\mathbb{R}^n} |\widehat{f}|^2 \, d\xi.$$

Extension: Since A is dense in L^2, this gives us that f extends to a linear operator $\tilde{\mathcal{F}} : L^2 \to L^2$ such that $\|\tilde{\mathcal{F}}\|_2 = \|f\|_2$.

It remains to check that $\tilde{\mathcal{F}} = \mathcal{F}(f)$ for $f \in L^1 \cap L^2$. Set

$$\rho(x) = e^{-\pi|x|^2}, \qquad \rho_r(x) = \frac{1}{t^n}\rho(x/t).$$

Let $f \in L^1 \cap L^2$. We have $\rho_t * f \in L^1 \cap L^2$, and

$$\widehat{\rho_t * f} = \widehat{\rho}_t \widehat{f} = \underbrace{e^{2\pi i|\xi|^2}}_{\in L^1} \underbrace{f(\xi)}_{\in L^\infty}.$$

So $\widehat{\rho}_t * f \in L^1$. This means that $\rho_t * f \in A$. We have that

$$\|\mathcal{F}(\rho_t * f) - \mathcal{F}(f)\|_2 = \|\tilde{\mathcal{F}}(\rho_t * f) - \tilde{\mathcal{F}}(f)\|_2 = \|\rho_t * f - f\|_2,$$

$$\|\mathcal{F}(\rho_t * f) - \tilde{\mathcal{F}}(f)\|_\infty \le \|\rho_r * f - f\|_1.$$

Let $B \subseteq \mathbb{R}^n$ be a bounded ball. We have

$$\|\tilde{\mathcal{F}}(f) - \mathcal{F}(f)\|_2 \le \|\tilde{\mathcal{F}}(f) - \mathcal{F}(\rho_t * \mathcal{F})\|_2 + \|\mathcal{F}(\rho_t * f) - \mathcal{F}(f)\|_{L^2(B)} \le \|\tilde{\mathcal{F}}(f) - \mathcal{F}(\rho_t * \mathcal{F})\|_2 + \|\mathcal{F}(\rho_t * f) - \mathcal{F}(f)\|_\infty |B|$$

So we conclude that $\tilde{\mathcal{F}}(f) = \mathcal{F}(f)$ a.e. on B. □

Corollary 21.3. *For $1 \le p \le 2$ and $q = p/(p-1)$, we obtain an extension to $\mathcal{F} : L^p \to L^q$ such that $\|\mathcal{F}(f)\|_q \le \|f\|_p$.*

21.3 Producing periodic functions from L^1 functions

Theorem 21.2. *Let $f \in L^1$.*

1. *There exists a periodic $Pf : \mathbb{R}^n \to \mathbb{R}$ such that $\|Pf\|_1 \le \|f\|_1$.*
2. *$\widehat{Pf}^{\mathbb{T}^n}(\ell) = \widehat{f}^{\mathbb{R}^n}(\ell)$.*
3. *$Pf(x) = \sum_{k \in \mathbb{Z}^n} \tau_k f(x)$.*

Proof. Let $Q = [-1/2, 1/2)^n$. Set $F_m(x) = \sum_{|k| \le m, k \in \mathbb{Z}^n} f(x-k)$. By the monotone convergence theorem,

$$\int_Q \sum_{k \in \mathbb{Z}^n} |f(x-k)|\, dx = \sum_{k \in \mathbb{Z}^n} \int_Q |f(x-k)|\, dx = \sum_{k \in \mathbb{Z}^n} \int_{Q+k} |f(x)|\, dz = \int_{\mathbb{R}^n} |f(z)|\, dz.$$

This proves that the series $(F_m(x))_m$ converges absolutely for a.e. $x \in Q$. So $(F_m(x))_m$ converges for a.e. $x \in Q$ to a value $Pf(x)$. We have that Pf is periodic. We also get that

$$\|Pf\|_{L^1(Q)} \le \|f\|_1.$$

This completes the proofs of the first and third statements.

If $\ell \in \mathbb{Z}^n$, then

$$\widehat{Pf}^{\mathbb{T}^n}(\ell) = \int_Q Pf(x) e^{-2\pi i \ell \cdot x}\, dx$$

$$= \int_Q \sum_Q \sum_{k \in \mathbb{Z}^n} f(x-k) e^{-2\pi i \ell \cdot x}\, dx$$

Let $z = x - k$.

$$= \sum_{k \in \mathbb{Z}^n} \int_{Q+k} f(z) e^{-2\pi i \ell \cdot z} e^{-2\pi i k \cdot \ell}\, dz$$

$$= \int_{\mathbb{R}^n} f(z) e^{2\pi i \ell \cdot z}\, dz = \widehat{f}(\ell). \qquad \square$$

22 The Poisson Summation Formula and Integrability of the Fourier Transform

This lecture was given by a guest lecturer.

22.1 The Poisson summation formula

Recall that if $E_k(x) =^{2\pi i k\cdot x}$, then $\{E_k : k \in \mathbb{Z}^n\}$ is an orthonormal basis of $L^2(\mathbb{T}^n)$. We have also shown the following:

Theorem 22.1. *If $f \in L^1(\mathbb{R}^n)$, then the series $\sum_{k\in\mathbb{Z}^N} \tau_k f$ converges pointwise a.e. and in $L^1(\mathbb{T}^n)$ to a function Pf such that $\|Pf\|_1 \leq \|f\|_1$. Moreover, $\widehat{Pf}(k) = \widehat{f}(k)$.*

We have also shown the following theorem in the $\mathbb{R}^n$ case, but here is the form of the theorem in the $\mathbb{T}^n$ case.

Theorem 22.2 (Hausdorff-Young inequality)**.** *Suppose that $1 \leq p \leq 2$ and q is the the conjugate exponent of p. If $f \in L^p(\mathbb{T}^n)$, then $\widehat{f} \in \ell^q(\mathbb{Z}^n)$, and $\|\widehat{f}\|_{\ell^q(\mathbb{Z}^n)} \leq \|f\|_{L^p(\mathbb{T}^n)}$.*

Theorem 22.3 (Poisson summation formula)**.** *Suppose that $f \in C(\mathbb{R}^n)$ satisfies $|f(x)| \leq C/(1+|x|)^{n+\varepsilon}$ and $|\widehat{f}(\xi)| \leq C/(1+|\xi|)^{n+\varepsilon}$ for some $C, \varepsilon > 0$. Then*

$$\sum_{k\in\mathbb{Z}} f(x+k) = \sum_{k\in\mathbb{Z}^n} \widehat{f}(k) e^{2\pi i l\cdot x},$$

where both series converge absolutely and uniformly on $\mathbb{T}^n$. In particular,

$$\sum_{k\in\mathbb{Z}^n} f(k) = \sum_{k\in\mathbb{Z}^n} \widehat{f}(k).$$

Proof. Since $|f(x)| \leq C/(1+|x|)^{n+\varepsilon}$, for all $x \in \mathbb{T}^n$,

$$|f(x+k)| \leq \frac{C}{(1+|x+k|)^{n+\varepsilon}} \leq \frac{C'}{(1+|k|)^{n+\varepsilon}}.$$

Then compare

$$\sum_{k\in\mathbb{Z}} \frac{C}{(1+|k|)^{n+\varepsilon}} \sim \int_{\mathbb{R}^n} \frac{1}{(1+|x|)^{n+\varepsilon}}\, dx.$$

This implies that

$$\sum_{k\in\mathbb{Z}^n} f(x+k) \overset{C(\mathbb{T}^n)}{=} Pf(x)$$

for all $x \in \mathbb{T}^n$.

By the previous theorem, we have $Pf \in L^1(\mathbb{T}^n)$ and $\widehat{Pf}(k) = \widehat{f}(k)$. Then $Pf \in L^2(\mathbb{T}^n)$, and since $\{E_k : k \in \mathbb{Z}^n\}$ is an orthonormal basis of $L^2(\mathbb{T}^n)$, we have

$$Pf \overset{L^2(\mathbb{T}^n)}{=} \sum_{k\in\mathbb{Z}^n} \widehat{Pf} e^{2\pi i k\cdot x} = \sum_{k\in\mathbb{Z}^n} \widehat{f}(k) e^{2\pi i k\cdot x}.$$

By the decay of $\widehat{f}$, $Pf(x) \overset{C(\mathbb{T}^n)}{=} \sum_{k\in\mathbb{Z}^n} \widehat{f}(k) e^{2\pi i k\cdot x}$. □

22.2 Integrability of the Fourier transform

The Fourier inversion theorem shows how to use $\widehat{f}$ to represent f is $f, \widehat{f} \in L^1(\mathbb{R}^n)$. In $\mathbb{T}^n$, if $f \in L^1(\mathbb{T}^n)$ and $\widehat{f} \in \ell^1(\mathbb{Z}^n)$, then the Fourier series

$$\sum_{k\in\mathbb{Z}^n} \widehat{f}(k) e^{2\pi i k\cdot x}$$

converges absolutely and uniformly to a function g. Since $\ell^1 \subseteq \ell^2$, it follows that $f \in L^2$ and the serires converges to f in L^2. Hence, $f = g$ a.e. We have 2 questions:

1. Under what conditions is $\widehat{f}$ integrable?
2. How can f be recovered from $\widehat{f}$ if $\widehat{f}$ is not integrable?

Theorem 22.4. *Suppose that f is periodic and absolutely continuous on $\mathbb{R}$, and $f' \in L^p(\mathbb{T})$ for some $p > 1$. Then $\widehat{f} \in \ell^1(\mathbb{Z})$.*

Proof. By integration by parts, $\widehat{f'}(k) = 2\pi i k \widehat{f}(k)$. Hence, by Hölder's inequality,

$$\begin{aligned}
\sum_{k\neq 0} |\widehat{f}(k)| &\le \underbrace{\left(\sum_k (2\pi|k|)^{-p}\right)^{1/p}}_{=:C_p} \left[\sum_{k\neq 0} (2\pi|k\widehat{f}(k)|^q)^{1/q}\right] \\
&= C_p \left(\sum_{k\neq 0} |\widehat{f'}(k)|^q\right)^{1/p} \\
&\le C_p \|\widehat{f'}\|_{\ell^q(\mathbb{Z})}.
\end{aligned}$$

Since $L^p(\mathbb{T}) \subseteq L^2(\mathbb{T})$ for $p > 2$, we can assume that $1 < p \le 2$. By the Hausdorff-Young inequality,

$$\sum_{k\neq 0} |\widehat{f}(k)| \le C_p \|f'\|_{L^p(\mathbb{T})}.$$

Adding $|\widehat{f}(0)|$ to both sides, we see that

$$\|\widehat{f}\|_{\ell^1(\mathbb{Z})} < \infty.$$ □

Lemma 22.1. *If $f, g \in L^2(\mathbb{R}^n)$, then $(\widehat{f}\widehat{g})^\vee = f * g$.*

Proof. By assumption, we know that $\widehat{f}, \widehat{g} \in L^2(\mathbb{R}^n)$. Then $\widehat{f}\widehat{g} \in L^1(\mathbb{R}^n)$. So $(\widehat{f}\widehat{g})^\vee$ makes sense. So for $x \in \mathbb{R}^n$, define $h(y) = \overline{g(x-y)}$. Then $\widehat{h}(\xi) = \overline{\widehat{g}(\xi)}e^{-2\pi i\xi\cdot x}$. Then

$$f * g(x) = \int_{\mathbb{R}^n} f\overline{h} = \int \widehat{f}\overline{\widehat{h}} = \int \widehat{f}(\xi)\widehat{g}(\xi e^{2\pi i\xi\cdot x}\,d\xi = (\widehat{f}\widehat{g})^\vee. \qquad \square$$

23 Recovering Functions From Their Fourier Series

This lecture was given by a guest lecturer.

23.1 Recovering functions from their Fourier series

Theorem 23.1. *Suppose that $\Phi \in C(\mathbb{R}^n)$ satisfies $|\Phi(\xi)| \leq C(1+|\xi|)^{-n-\varepsilon}$, $|\Phi^\vee(x)| \leq C(1+|x|)^{-n-\varepsilon}$, and $\Phi(0) = 1$. Given $f \in L^1(\mathbb{T}^n)$, for any $t > 0$, set*

$$f^t(x) = \sum_{k \in \mathbb{Z}^n} \widehat{f}(k)\Phi(tz)e^{2\pi i k\cdot x}.$$

1. *If $f \in L^p(\mathbb{T}^n)$, then $\|f^t - f\|_p \to 0$ as $t \to 0$. If $f \in C(\mathbb{T}^n)$, then $f^t \to f$ uniformly as $t \to 0$.*
2. *$f^t(x) \to f(x)$ for every x in the Lebesgue set of f.*

Proof. First, let $\phi = \Phi^\vee$, and let $\phi_t(x) = t^{-n}\phi(t^{-1}x)$. Then $\widehat{\phi_t}(\xi) = \Phi(t\xi)$. Since $|\Phi(\xi)| \leq C(1+|\xi|)^{-n-\varepsilon}$, we have $\Phi \in L^1(\mathbb{R}^n)$. So $\phi \in C(\mathbb{R}^n)$. And, moreover,

$$\phi_t(x) = t^{-n}\phi(t^{-1}x) \leq Ct^{-n}(1+|t^{-1}x|)^{-n-\varepsilon} \leq Ct^{-n}(1+|x|)^{-n-\varepsilon},$$

where the last inequality holds for $t \ll 1$. Also,

$$\widehat{\phi_t}(\xi) = \Phi(t\xi) \leq C(1+t|\xi|)^{-n-\varepsilon} \overset{0<t<1}{\leq} C(t+t|\xi|)^{-n-\varepsilon} = Ct^{-n-\varepsilon}(1+|\xi|)^{-n-\varepsilon}.$$

Applying the Poisson summation formula for each fixed t, we get

$$\sum_{k\in\mathbb{Z}^n} \varphi_t(x-k) = \sum_{k\in\mathbb{Z}^n} \widehat{\phi_t}(k)e^{2\pi i k\cdot x} = \sum_{k\in\mathbb{Z}^n} \Phi(tk)e^{2\pi i k\cdot x} =: \psi_t(x) \in L^2(\mathbb{T}^n) \subseteq L^1(\mathbb{T}^n).$$

Then $\widehat{f * \psi_t}(k) = \widehat{f}(k)\widehat{\psi_t}(k)$, as $f, \psi_t \in L^1$ for each t. As $\psi_t \in L^2$, we have that $\psi_t(x) = \sum_{k\in\mathbb{Z}^n} \widehat{\psi_t} e^{2\pi k\cdot x}$, which means that $\widehat{\psi_t}(k) = \Phi(tk)$ (since the Fourier series coefficients agree). So

$$\widehat{f * \psi_t}(k) = \widehat{f}(k)\Phi(tk) = \widehat{f^t}(k).$$

So we get $f^t = f * \psi_t$ by taking the inverse Fourier transform. Hence, for all $1 \leq p \leq \infty$, by Young's inequality (and a theorem we have already proven),

$$\|f^t\|_p = \|f * \psi_t\|_p \leq \|f\|_p\|\psi_t\|_1 \leq \|f\|_p\|\phi_t\|_1 = \|f\|_p\|\phi\|_1.$$

So the operator $f \to f^t$ is uniformly bounded in L^p for $1 \leq p \leq \infty$.

Notice that Φ is continuous and $\Phi(0) = 1$. We have $f^t \to f$ uniformly if f is a trigonometric polynomial, i.e. $\widehat{f}(k) = 0$ for all but finitely many k: $f = \sum_{j=1}^m \widehat{f}(k_j)e^{2\pi i k_j\cdot x}$. By the Stone-Weierstrass theorem, the trigonometric polynomials are dense in $C(\mathbb{T}^n)$ and hence also dense in $L^p(\mathbb{T}^n)$. So for all $\varepsilon > 0$, there exists a trigonometric polynomial f_n such that $\|f - f_n\|_p \leq \varepsilon$. Then

$$\begin{aligned}\|f^t - f\|_p &\leq \|f^t - f_n^t\| + \|f_n^t - f_n\|_p + \|f_n - f\|_p \\ &\leq \|\phi\|_1 \|f - f_n\|_p + \|f_n^t - f_n\|_p + \|f_n - f\|_p \\ &\leq (\|\phi\|_1 + 1)\varepsilon.\end{aligned}$$

This proves the first statement.

For the second statement, without loss of generality, assume that 0 is a Lebesgue point of f. With $Q = [-1/2, 1/2)^n$, we have

$$f^t(0) = f * \psi_t(0) = \int_Q f(x)\psi_t(-x)\,dx = \int_Q f(x)\phi_t(-x)\,dx + \sum_{k\neq 0} \int_Q f(x)\phi_t(-x_k)\,dx$$

Since

$$|\phi_t(x)| \leq Ct^{-n}(1 + t^{-1}|x|)^{-n-\varepsilon} = Ct^{\varepsilon}(t + |x|)^{-n-\varepsilon} \leq Ct^{\varepsilon}|x|^{-n+\varepsilon},$$

we have

$$|\phi_t(x + k)| \leq Ct^{\varepsilon}|-x + k|^{-n-\varepsilon} \leq Ct^{\varepsilon}\left|\frac{k}{2}\right|^{-n-\varepsilon} = C2^{n+\varepsilon}t^{\varepsilon}|k|^{-n-\varepsilon}$$

for $k \neq 0$. So we get

$$\sum_{k\neq 0}\left|\int f(x)\phi_t(-x + k)\right| \leq \left[C2^{n+\varepsilon}\|f\|_1 \sum_{k\neq 0}|k|^{-n-\varepsilon}\right] t^{\varepsilon} \xrightarrow{t\to 0} 0.$$

On the other hand, if we define $g = f\mathbb{1}_Q \in L^1(\mathbb{R}^n)$,

$$\lim_{t\to 0}\int_Q f(x)\phi_t(-x)\,dx = \lim_{t\to 0} g * \phi_t(0) = g(0) = f(0).$$

So we get that $f^t(0) \to f(0)$. $\square$

24 Distributions and Smooth Urysohn's Lemma

24.1 Distributions

Throughout this section, $U \subseteq \mathbb{R}^n$ is an open set.

Definition 24.1. If $E \subseteq \mathbb{R}^n$, $C_c^\infty(E)$ is the set of $\phi \in C_c^\infty(\mathbb{R}^n)$ such that $\operatorname{supp}(\phi) \subseteq E$.

We endow $C_c^\infty(U)$ with the following topology: $(\phi_j)_{j\in\mathbb{N}} \subseteq C_c^\infty(U)$ converges to $\phi \in C_c^\infty(U)$ if there exists a compact $K \subseteq U$ such that

- $\operatorname{supp}(\phi_j) \subseteq K$ for all j,
- $\partial^\alpha \phi_j \to \partial^\alpha \phi$ uniformly on K for all $\alpha \in \mathbb{N}^n$.

Definition 24.2. Let X be a locally convex topological vector space. A linear operator $T : C_c^\infty(U) \to X$ is **continuous** if for each compact $K \subseteq U$, $T|_{C_c^\infty(K)}$ is continuous.

Definition 24.3. Let U' be an open subset of $\mathbb{R}^n$. A linear operator $T : C_c^\infty(U) \to C_c^\infty(U')$ is **continuous** if for each compact $K \subseteq U$, there exists a compact $K' \subseteq U'$ such that $T(C_c^\infty(K)) \subseteq C_c^\infty(K')$, and $T : C_c^\infty(K) \to C_c^\infty(K')$ is continuous.

Definition 24.4. If $T : C_c^\infty(U) \to \mathbb{R}$ is linear and continuous, we say that T is a **distribution** on U and write $T \in \mathcal{D}'(U)$.[2]

Definition 24.5. If $V \subseteq U$ and $T, S \in \mathcal{D}'(U)$, we say that $T = S$ on V if $T(\phi) = S(\phi)$ for all $\phi \in C_c^\infty(V)$.

Definition 24.6. A sequence $(T_j)_{j\in\mathbb{N}} \subseteq \mathcal{D}'(U)$ **converges** to $T \in \mathcal{D}'$ if $\lim_{j\to\infty} T_j(\phi) = T(\phi)$ for all $\phi \in C_c^\infty(U)$.

That is, $\mathcal{D}'(U)$ is endowed with the weak* topology.

Example 24.1. Let $f \in L^1_{\text{loc}}(U)$. Define

$$T(\phi) = \int_U f\phi\, dx, \qquad \phi \in C_c^\infty(U).$$

This is a distribution.

Example 24.2. Let μ be a Radon measure on U. Define

$$T(\phi) = \int_U \phi(x)\, d\mu(x).$$

For example, let $x_0 \in U$, and $\mu = a\delta_{x_0}$. Set

$$T(\phi) = a\phi(x_0) = \int_U \phi(x)\, d\mu(x).$$

This is a distribution.

[2] This notation is because some people call $\mathcal{D} := C_c^\infty(U)$ and denote the dual by $'$.

Notation: If $\phi : \mathbb{R}^n \to \mathbb{R}$, set $\tilde{\phi}(x) = \phi(-x)$.

Proposition 24.1. *Let $f \in L^1(\mathbb{R}^n)$. For each $t > 0$, set $f_t(x) = t^{-n}\phi(x/t)$ for $x \in \mathbb{R}^n$. Assume that $\int_{\mathbb{R}^n} f(x)\,dx = 1$. Define*

$$T_t(\phi) = \int_{\mathbb{R}^n} f_t(x)\phi(x)\,dx.$$

Then $T_t \to \delta_0$ in $\mathcal{D}'(\mathbb{R}^n)$; that is, $T_t \to T_0$, where $T_0 = \delta_0$.

Remark 24.1. Often, people will view f_t as its distribution T_t and call the distribution f_t.

Proof. Let $\phi \in C_c^\infty(\mathbb{R}^n)$. Observe that

$$T_t(\phi) = \int_{\mathbb{R}^n} f_t(x)\tilde{\phi}(0 - x)\,dx = f_t * \phi(0).$$

So we have

$$\lim_{t\to 0} T_t(\phi) = \lim_{t\to 0} f_t * \tilde{\phi}(0) = \tilde{\phi}(0) = \phi(0).$$ □

24.2 Smooth Uryson's Lemma

Proposition 24.2 (extension of Urysohn's lemma)**.** *Let $K \subseteq \mathbb{R}^n$ be compact, and let $U \subseteq \mathbb{R}^n$ be an open set containing K. Then there exists $\phi \in C_c^\infty(\mathbb{R}^n, [0,1])$ such that $\phi|_K = 1$ and* $\operatorname{supp}(\phi) \subseteq U$*.*

Remark 24.2. Urysohn's lemma is the case where we do not assume that ϕ is smooth.

Proof. Let $\rho \in C_c^\infty(\mathbb{R}^n)$ be such that $\rho \geq 0$, $\operatorname{supp}(\rho) \subseteq \overline{B_1(0)}$ and $\int_{\mathbb{R}^n} \rho(x)\,dx = 1$. Set $\rho_t(x) = t^{-n}\rho(x/t)$ for $t > 0$ and $x \in \mathbb{R}^n$. By Urysohn's lemma, there is a $g \in C_c(\mathbb{R}^n, [0,1])$ such that $g|_{K_\varepsilon} = 1$, $\operatorname{supp}(g) \subseteq U_\varepsilon$, where $K_\varepsilon = \{x \in \mathbb{R}^n : \operatorname{dist}(x, K) \leq \varepsilon\}$ and $U_\varepsilon = \{x \in U : \operatorname{dist}(x, U^c) > \varepsilon\}$. As K is compact, let $\delta = \operatorname{dist}(K, U^c) > 0$. If $0 < \varepsilon < \delta$¡ then $K \subseteq U_\varepsilon$, K_ε is compact, and U_ε is open. Let $\phi = \rho_{\delta/4} * g$, and let $\varepsilon = \delta/4$. Since $\rho_{\delta/4} \in C_\infty(\mathbb{R}^n)$, we have $\phi \in C^\infty(U)$. Note that

$$\phi(x) = \int_{\mathbb{R}^n} \rho(y/\varepsilon)\frac{1}{\varepsilon^n} g(x-y)\,dy = \int_{B_\varepsilon(0)} \rho_\varepsilon(x) g(x-y)\,dy.$$

If $x \in K$ and $|y| < \varepsilon$¡ the $x - y \in K_\varepsilon$, and so $g(x-y) = 1$. Hence,

$$\phi(x) = \int_{B_\varepsilon(0)} \rho_\varepsilon(x)\,dx = 1.$$

If $x \notin U^\varepsilon$, then $g(x-y) = 0$ if $|y| < \varepsilon$. Hence, $\phi(x) = 0$. □

25 Extensions and Transformations of Distributions

25.1 Extension of distributions

Let $U \subseteq \mathbb{R}^n$ be open. If $V \subseteq U$ is open and $T, S \in \mathcal{D}'(U)$, we say that $T = S$ on V if $T|_{C_c^\infty(V)} = S|_{C_c^\infty(V)}$. Assume $V_1, V_2 \subseteq U$ are open and $T, S \in \mathcal{D}'(U)$ are such that $T|_{C_c^\infty(V_1)} = S|_{C_c^\infty(V_1)}$ and $T|_{C_c^\infty(V_2)} = S|_{C_c^\infty(V_2)}$.We want to show that $T|_{C_c^\infty(V_1 \cup V_2)} = S|_{C_c^\infty(V_1 \cup V_2)}$.

Here is a wrong proof: Let $\phi \in C_c^\infty(V)$, and assume that $V_1 \cap V_2 = \varnothing$. Then

$$T(\varphi) = T(\mathbb{1}_{V_1}\phi + \mathbb{1}_{V_2}\phi) = T(\mathbb{1}_{V_1}\phi) + T(\mathbb{1}_{V_2}\phi) = S(\mathbb{1}_{V_1}\phi) + S(\mathbb{1}_{V_2}\phi) = S(\varphi).$$

This is not a correct proof because $\mathbb{1}_{V_1}\phi$ need not be in C_c^∞.

Theorem 25.1. *Let $(V_\alpha)_{\alpha \in I}$ be open subsets of U and let $V = \bigcup_{\alpha \in I} V_\alpha$. Let $T, S \in \mathcal{D}'(V)$ be such that $T|_{C_c^\infty(V_\alpha)} = S|_{C_c^\infty(V_\alpha)}$ for all $\alpha \in I$. Then $T|_{C_c^\infty(V)} = S|_{C_c^\infty(V)}$.*

Proof. Let $\phi \in C_c^\infty(V)$. We are to show that $T(\phi) = S(\phi)$. Set $K = \operatorname{supp}(\phi) \subseteq V = \bigcup_{\alpha \in I} V_\alpha$. Since K is compact, there are $\alpha_1, \ldots, \alpha_m \in I$ such that $K \subseteq \bigcup_{j=1}^m V_{\alpha_j}$. For each $x \in K$, there exist $r(x) > 0$ and $j \in \{1, \ldots, m\}$ such that $B_{2r(x)}(x) \subseteq V_{\alpha_j}$. Note that $K \subseteq \bigcup_{x \in K} B_{r(x)}(x)$, and so there exists $x_1, \ldots, x_\ell \in K$ such that $K \subseteq \bigcup_{i=1}^\ell B_{r(x_i)}(x_i)$.

For each $j \in \{1, \ldots, m\}$, set $I_j = \{i \in \{1, \ldots, \ell\} : B_{2r(x_i)}(x_i) \subseteq V_{\alpha_j}$. Note that the set $K_j := \bigcup_{i \in I_j} \overline{B_{r(x_i)}(x_i)}$ is compact, and $K_j \subseteq V_{\alpha_j}$. By the extended Urysohn's lemma, there exists $f_j \in C_c^\infty(\mathbb{R}^n, [0,1])$ such that $f_j|_{K_j} \equiv 1$ and $\operatorname{supp}(f_j) \subseteq V_{\alpha_j}$. Set $\mathcal{E} = \{\sum_{j=1}^m f_j > 0\}$. On K, $\sum_{j=1}^m f_j \geq 1$, and so $K \subseteq \mathcal{E}$. We apply the extended Urysohn's lemma once more to obtain $f \in C_c^\infty(\mathbb{R}^n)$ such that $f|_K \equiv 1$ and $\operatorname{supp}(f) \subseteq \mathcal{E}$. Set $f_{m+1} = 1 - f$. Now $f_1 + \cdots + f_{m+1}$ is always strictly positive because $f_1 + \cdots + f_m > 0$ on $\mathcal{E}$ and 1 outside $\mathcal{E}$.

We can hence define

$$h_j = \frac{f_j}{\sum_{i=1}^{m+1} f_i} \in C_c^\infty(\mathbb{R}^n, [0,1])$$

for each $1 \leq j \leq m$. Note that $\operatorname{supp}(h_j) \subseteq V_{\alpha_j}$ and that $(\sum_{j=1}^m h_j)|_K \equiv 1$. Thus, $\phi = \phi \sum_{j=1}^m h_j$, so

$$T(\phi) = T\left(\phi \sum_{j=1}^m h_j\right) = \sum_{j=1}^m T(\phi h_j) = \sum_{j=1}^m S(\phi h_j) = S\left(\phi \sum_{j=1}^m h_j\right) = S(\phi). \qquad \square$$

25.2 Transformations of distirbutions

Definition 25.1. Let $T \in \mathcal{D}'(U)$. If $\alpha \in \mathbb{N}^n$ is a multi-index, define $\partial^\alpha T : C_c^\infty(U) \to \mathbb{R}$ as

$$(\partial^\alpha)T(\phi) = (-1)^{|\alpha|} T(\partial^\alpha \phi).$$

Definition 25.2. If $\psi \in C^\infty(U)$ and $T \in \mathcal{D}'(U)$, define $\psi T : C_c^\infty(U) \to \mathbb{R}$ as

$$(\psi T)(\phi) = T(\psi\phi), \qquad \phi \in C_c^\infty(U).$$

Definition 25.3. If $y \in \mathbb{R}^n$, we define $\tau_y(T) : C_c^\infty(U - y) \to \mathbb{R}$ as

$$\tau_y(T)(\phi) = T(\tau_{-y}\phi), \qquad \phi \in C_c^\infty(U).$$

Definition 25.4. Let $S : \mathbb{R}^N \to \mathbb{R}^n$ be a linear bijection, and set $V = S^{-1}(U)$. We define $T \circ S : C_c^\infty(V) \to \mathbb{R}$ as

$$T \circ S(\phi) = \frac{1}{|\det(S)|} T(\phi \circ S^{-1}), \qquad \phi \in C_c^\infty(V).$$

Theorem 25.2. *Let T, S, ψ, y, α be as above. Then*

1. $\partial^\alpha T, \psi T \in \mathcal{D}'(U)$
2. $\tau_y(T) \in \mathcal{D}'(U - y)$
3. $T \circ S \in \mathcal{D}'(V)$.

Proof. For the second statement, the idea is that τ_y is an isometry of $C_c^\infty(U) \to C_c^\infty(U-y)$.

For the third statement, the idea is that $|\det(S)|^{-1}\phi \circ S^{-1}$ is an isomorphism of $C_c^\infty(V)$ into $C_c^\infty(U)$.

For the first statement, let's take something weaker, say $g \in L^p(U)$. Then $\partial_{x_i} g$ exists as a distribution. Can we represent $\partial_{x_i g}$ as an L^p function? If we can, say $g \in W^{1,p}(U)$. Similarly, if $\partial^2_{x_I, x_j} g \in L^p(U)$, then say that $g \in W^{2,p}$. $\square$

We will continue this next time. This will involve **Sobolev spaces**.

26 Introduction to Sobolev Spaces

26.1 Sobolev spaces and uniqueness of distributional derivatives

Throughout this section, $\Omega \subseteq \mathbb{R}^d$ is a nonempty, open set.

Proposition 26.1. *Let $f \in L^1_{\mathrm{loc}}(\Omega)$ be such that $\int_\Omega f\phi\, dx = 0$ for all $\phi \in C_c^\infty(\Omega)$. Then $f \equiv 0$ a.e.*

Proof. Let $\rho \in C_c^\infty(\mathbb{R}^d)$ be such that $\rho \geq 0$, $\int_{\mathbb{R}^d} \rho\, dx = 1$, and $\mathrm{supp}(\rho) = B_1(0)$. Set $\rho_\varepsilon(x) = \varepsilon^{-d}\rho(x/\varepsilon)$. Let $x \in U$, and let $0\varepsilon_0 < \mathrm{dist}(x, \partial\Omega)$. Then

$$\rho_\varepsilon * f(x) = \int_{B_\varepsilon(x)} \rho_\varepsilon(x-y) f(y)\, dy = 0, \qquad 0 < \varepsilon < \varepsilon_0.$$

Thus, for almost every x,

$$0 = f(x) = \lim_{\varepsilon \to 0} \rho_\varepsilon * f(x).$$

□

Definition 26.1. Let $1 \leq p \leq \infty$, and let $m \in \mathbb{N}$. We say that $f \in W^{p,m}_{\mathrm{loc}}(\Omega)$ if $f \in L^p_{\mathrm{loc}}(\Omega)$ and if for every multi-index $\alpha \in \mathbb{N}^n$ such that $|\alpha| \leq m$, there exists $g_\alpha \in L^p_{\mathrm{loc}}(\Omega)$ such that

$$\int_\Omega f \partial^\alpha \phi\, dx = (-1)^{|\alpha|} \int_\Omega g_\alpha \phi\, dx \qquad \forall \phi \in C_c^\infty(\Omega).$$

In other words, the distributional derivative $\partial^\alpha f \in L^p_{\mathrm{loc}}$. When $f \in L^p(\Omega)$ and $g_\alpha \in L^p(\Omega)$ for $|\alpha| \leq m$, we write $f \in W^{m,p}(\Omega)$.

Remark 26.1. Thanks to the previous proposition, when g_α exists, it is uniquely determined a.e.

26.2 Translation of distributions

Notation: Let $\phi \in C_c^\infty(\Omega)$, and let $y \in \mathbb{R}^d$. We set $\phi_y(x) = \phi(x-y) = (\tau_y \phi)(x)$. Note that $\mathrm{supp}(\varphi_y) = \mathrm{supp}(\phi) + y$. Set

$$O_\phi = \{y \in \mathbb{R}^d : y + \mathrm{supp}(\phi) \subseteq \Omega\} = \{y \in \mathbb{R}^d : \mathrm{supp}(\phi_y) \subseteq \Omega\}.$$

Proposition 26.2. *O_ϕ is open and nonempty.*

Proof. Let $y \in O_\phi$, and set $\delta = \mathrm{dist}(y + \mathrm{supp}(\phi), \Omega^c) > 0$. If $y \in O_\phi$, then $B_{\delta/2}(y) \subseteq O_\phi$. Hence, O_ϕ is open. $O_\phi \neq \varnothing$ because $0 \in O_\phi$. □

Proposition 26.3. *If $T \in \mathcal{D}'(\Omega)$, $y \mapsto T(\phi_y)$ is continuous.*

Proof. Let $(y_n)_n \subseteq O_\phi$ be a sequence converging to y. We are to show that $\lim_n T(\phi_{y_n}) = T(\phi_y)$. Note that

$$\phi_{y_n}(x) = \phi(x - y_n) = \phi(x-y) - \int_0^1 \nabla\phi(x - y + t(y_n - y)) \cdot (y_n - y)\, dt.$$

This gives us that $(\phi_{y_n})_n$ converges to ϕ_y in C_c^∞. Indeed,

$$|\partial^\alpha \phi_{y-N} - \partial^\alpha \phi_y| \leq \|\nabla \partial^\alpha \phi\|_\infty \|y_n - y\|.$$

Since T is continuous, we conclude that

$$\lim_n T(\phi_{y_n}) = T(\phi_y). \qquad \square$$

Theorem 26.1. *Let $\phi \in C_c^\infty(\Omega)$, and let $T \in \mathcal{D}'(\Omega)$. Set $f(y) = T(\phi_y)$ for $y \in O_\phi$.*

1. *$f \in C^\infty(O_\phi)$, and*
$$D^\alpha f(y) = (-1)^{|\alpha|} T((D^\alpha \phi)_y).$$
2. *If $\psi \in L^1(O_\phi)$ has compact support, then*
$$T(\psi * \phi) = \int_{O_\phi} \psi(y) f(y)\, dy.$$

Proof. One proves by induction on α that $\partial^\alpha f$ exists, is continuous, and satisfies the equation. Assume $|\alpha| = 1$. Let $e_1, \dots, e_d$ be the standard basis of $\mathbb{R}^n$. We have for $t \in \mathbb{R}$

$$\phi_{y+te_i}(x) = \phi(x - y - te_i) = \phi(x-y) - \int_0^1 \partial_i \phi(x - y - t\tau e_i)\, d\tau.$$

Hence,

$$\frac{\phi_{y+te_i}(x) - \phi_y(x)}{t} = -\int_0^1 \partial_i \phi(x - y - t\tau e_i)\, d\tau.$$

In fact, we have

$$\frac{\partial^\alpha \phi_{y+te_i}(x) - \partial^\alpha \phi_y(x)}{t} = -\int_0^1 [\partial^\alpha \partial_i \phi(x - y - t\tau e_i) - \partial^\alpha \partial_i \phi(x-y)]\, d\tau - \partial^\alpha \partial_i \phi(x-y).$$

This shows that

$$\frac{\phi_y + te_u - \phi_y}{t}(x) \to -\partial_i \phi(x-y)$$

pointwise and in $C_c^\infty(\Omega)$. Hence,

$$\lim_{t\to 0} \frac{f(y+te_i) - f(y)}{t} = \lim_{t\to 0} \frac{T(\phi_y + te_i) - T(\phi_y)}{t} = \lim_{t\to\infty} T\left(\frac{\phi_{y+te_i} - \phi_y}{t}\right) = T(-(\partial\phi(x))_y).$$

Since $\partial_i \phi \in C_c^\infty(\Omega)$, by the previous proposition, $y \to T((\partial_i\phi)_y)$ is continuous. In conlcusion, f is continuously differentiable, and $\nabla d(y) = -T((\nabla\phi)_y)$. This concludes the proof of the first statement when $|\alpha| = 1$. By induction, we obtain the result for all α. $\square$

We will prove the second statement next time.

27 Applying Distributions to Convolutions

27.1 Uniform estimates of functions on bounded sets

Last time, we proved the first half this theorem:

Theorem 27.1. *Let $\phi \in C_c^\infty(\Omega)$, and let $T \in \mathcal{D}'(\Omega)$. Set $f(y) = T(\phi_y)$ for $y \in O_\phi$.*

1. *$f \in C^\infty(O_\phi)$, and*
$$D^\alpha f(y) = (-1)^{|\alpha|} T((D^\alpha \phi)_y).$$
2. *If $\psi \in L^1(O_\phi)$ has compact support, then*
$$T(\psi * \phi) = \int_{O_\phi} \psi(y) f(y)\, dy.$$

To prove the second half, we first make some remarks.

Remark 27.1. Fix $R > 0$, and set $Q = [-R, R]^d$. There are $a : (0, \infty) \to (0, \infty)$ and $m : (0, \infty) \to \mathbb{N}$ such that for all $\varepsilon > 0$,
$$\lim_{\varepsilon \downarrow 0} a(\varepsilon) = 0$$
and such that for every $\varepsilon > 0$, there is a partition $\{Q\}_{i=1}^{m(\varepsilon)}$ of squares of diameters less than $a(\varepsilon)$.

These conclusions extend to any set $\Omega \subseteq [-R, R]^d$ with $\Omega_i = Q_i \cap \Omega$.

Definition 27.1. Let $A \subseteq \mathbb{R}^d$, and let $f : A \to \mathbb{R}$. We define the **oscillation** of f as
$$\operatorname{osc}(f, A, \delta) = \sup_{x,y \in A} \{|f(x) - f(y)| : |x - y| \leq \delta\}.$$

Remark 27.2. Assume $A = \Omega$ and $f : \Omega \to \mathbb{R}$ is uniformly continuous. Then
$$\int_\Omega f(x)\, dx = \sum_{i=1}^{m(\varepsilon)} \int (f(x) - f(x_i))\, dx + |\Omega_i^\varepsilon| f(x_i^\varepsilon),$$
where $x_i^\varepsilon \in \Omega_i^\varepsilon$. As a consequence,
$$\left| \int_\Omega f(x)\, dx - \sum_{i=1}^{m(\varepsilon)} |\Omega_i^\varepsilon| f(x_i^\varepsilon) \right| \leq i |\Omega| \operatorname{osc}(f, \Omega, a(\varepsilon)).$$

Remark 27.3. If $\phi \in C_c^\infty(\Omega)$ and $T \in \mathcal{D}'(\Omega)$, we set
$$\phi_y(x) = \phi(x - y), \qquad x \in y + \operatorname{supp}(\phi),$$
and $y \mapsto T(\phi_y)$ is continuous on $O_\phi = \{y \in \mathbb{R}^d : y + \operatorname{supp}(\phi) \subseteq \Omega\}$.

27.2 Proof of the theorem

Now we can prove the theorem.

Proof. Let $\psi \in L^1(O_\phi)$ be such that $\operatorname{supp}(\psi) \subseteq O_\phi$, We are to show that

$$\int_{O_\phi} \psi(y) T(\phi_y)\, dy = T(\psi * \psi).$$

Case 1: $\psi \in C_c^\infty(O_\phi)$. Since $y \mapsto f(y) := \psi(y)T(\phi(y))$ is uniformly continuous on O_ϕ,

$$\left| \int_{O_\phi} \psi(y) T(\phi_y)\, dy - \sum_{i=1}^{m(\varepsilon)} \psi(y_i^\varepsilon) T(\phi_{y_i^\varepsilon}) |\Omega_i^\varepsilon| \right| \leq \operatorname{osc}(f, O_\phi, a(\varepsilon)) |O_\phi|$$

for some $y_i^\varepsilon \in \Omega_i^\varepsilon$ independent of T, ϕ, ψ. Set $\eta^\varepsilon(x) = \sum_{i=1}^{m(\varepsilon)} \psi(y_i^\varepsilon)\phi(x - y_i^\varepsilon)$. Let K_1 be the closure of the set $\bigcup_{y \in O_\phi}(y + \operatorname{supp}(\phi)) \subseteq \Omega$. Then K_1 is compact.

For any multi-index $\alpha \in \mathbb{N}^d$,

$$\partial^\alpha \eta^\varepsilon(x) = \sum_{i=1}^{m(\varepsilon)} \psi(y_i^\varepsilon) \partial^\alpha \phi(x - y_i^\varepsilon) |\Omega_i^\varepsilon|.$$

This converges to $\int_{O_\phi} \psi(y) \partial^\alpha \phi(x-y)\, dy = \psi * \partial^\alpha$ uniformly:

$$\left| \int_{O_\phi} \psi(y) \partial^\alpha \phi(x-y)\, dy - \partial^\alpha \eta^\varepsilon(x) \right| \leq |\Omega| \operatorname{osc}(g_\varepsilon^x, \Omega, a(\varepsilon)) \xrightarrow{\varepsilon \to 0} 0,$$

where $g_\varepsilon^x(y) = \psi(y)\partial^\alpha \phi(x-y)$. This means $(\eta_\varepsilon)_\varepsilon$ converges to $\psi * \phi$ in $C_c^\infty(O_\phi)$. Consequently,

$$T(\phi * \psi) = \lim_{\varepsilon \to 0} T(\eta_\varepsilon) = \lim_{\varepsilon \to 0} \sum_{i=1}^{m(\varepsilon)} |\Omega_i^\varepsilon| T(\psi_{y_i^\varepsilon}) = \int_{O_\phi} \psi(y) T(\phi_y)\, dy.$$

Case 2: $\psi \in L^1(\emptyset_\phi)$ and $\operatorname{supp}(\psi) \subseteq O_\phi$: For each $\delta > 0$, let $\psi_\delta \in C_c^\infty(\emptyset_\phi)$ be such that $\int_{O_\phi} |\psi - \psi_\delta|\, dx \leq \delta$, and assume there exists a compact K_2 such that $\operatorname{supp}(\psi_\delta) \subseteq K_2 \subseteq O_\phi$. Note that for a multi-index $\alpha \in \mathbb{N}^d$,

$$\partial_\alpha(\psi_\delta * \phi) = \partial^\alpha \phi * \psi_\delta \to \partial^\alpha \phi * \psi$$

uniformly on K_2. Hence, $\psi_\delta * \phi \to \psi * \phi$ uniformly as $\delta \to 0$. We conclude that

$$T(\psi * \phi) = \lim_{\delta \to 0} T(\psi_\delta * \phi) = \lim_{\delta \to 0} \int_{O_\phi} \psi_\delta(y) T(\phi_y)\, dy = \int_{O_\phi} \phi(y) T(\phi_y),$$

using the dominated convergence theorem. $\square$

Let $\phi \in C^1(\Omega$, and assume that $\int_\Omega |\phi|^p \, dx + \int_\Omega |\nabla\phi|^p \, dx < \infty$. Then $\nabla\phi$ as a distribution is equal to the usual $\nabla\phi$.

A consequence of our result will be that for every y and a.e. x,

$$\phi(x+y) - \phi(y) = \int_0^1 \nabla\phi(x+ty) \cdot y \, dt.$$

28 Distributions of Differences

28.1 Differences of functions in Sobolev spaces

Let $\Omega \subseteq \mathbb{R}^d$ be an open set. If $A \subseteq \mathbb{R}^d$, $f : A \to \mathbb{R}$¡ and $y \in \mathbb{R}^d$¡ we set $f_y(x) = f(x-y)$ for $x \in A + y$. If $\phi \in C_c^\infty(\Omega)$, let $O_\phi = \{y \in \mathbb{R}^d : y + \operatorname{supp}(\phi) \subseteq \Omega\}$.

Proposition 28.1. *Let $\phi \in C_c^\infty(\Omega)$ and $y \in \mathbb{R}^d$. Then $K = \bigcup_{y\in[0,1]}(ty + \operatorname{supp}(\phi))$ is compact.*

Proof. Set $f(t,z) = ty + z$. $f : \mathbb{R}^{d+1} \to \mathbb{R}^d$ is continuous, and $K = f([0,1] \times \operatorname{supp}(\phi))$ is compact as the image of a compact set by a continuous function. ☐

Theorem 28.1. *Let $T \in \mathcal{D}'(\Omega)$, and let $y \in \mathbb{R}^d$.*

1. *If $\phi \in C_c^\infty(\Omega)$ and $ty + \operatorname{supp}(\phi) \subseteq \Omega$ for all $t \in [0,1]$, then*

$$T(\phi_y) = T(\phi) = \int_0^1 \sum_{j=1}^d y_j \partial_j T(\phi_{ty})\, dt.$$

2. *If $f \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^d)$, then for a.e. $x \in \mathbb{R}^d$,*

$$f(x+y) - f(x) = \int_0^1 \nabla f(x+ty) \cdot y\, dt.$$

In the second case, if we could show that $\frac{d}{dt}T(\phi_{ty}) = \nabla T(\phi_{ty}) \cdot y$ and that this derivative is continuous, we could just use the fundamental theorem of calculus.

Proof. Set $K = \bigcup_{t\in[0,1]}(ty + \operatorname{supp}(\phi))$. Then $K \subseteq \Omega$ is compact. For $x \in \mathbb{R}^d$ and $h \neq 0$,

$$L_h(x) := \frac{\phi(x-(t+h)y) - \phi(x-ty)}{h} = -\int_0^1 \nabla\phi(x - ty - \tau hy) \cdot y\, d\tau.$$

Note that $L_h \in C_c^\infty(\Omega)$ if $0 < |h| \ll 1$, and

$$\lim_{h\to 0} H_h(x) = \nabla\phi(x-ty)\cdot y =: L_0(x).$$

Also, $(L_h)_h$ converges to L_0 in $C_c^\infty(\Omega)$. Thus,

$$\begin{aligned}\frac{d}{dt}T(\phi_{ty}) &= \lim_{h\to 0} T(L_h) = T(L_0) = T(-\nabla\phi(x-ty)\cdot y)\\ &= -\sum_{j=1}^d y_j T(\partial_j \phi(\cdot - ty)) = \sum_{j=1}^d y_j \partial_j T(\phi(\cdot - ty))\end{aligned}$$

$$= \sum_{j=1}^{d} y_j \partial_j T(\phi_{ty}).$$

As $t \to \partial_j T(\phi_{t,y})$ is continuous, we conclude that $t \to \frac{d}{dt} T(\phi_{ty})$ is continuous. So we get

$$T(\phi_y) - T(\phi) = \int_0^1 \frac{d}{dt}(T(\phi_{ty}))\, dt = \int_0^1 \nabla T(\phi_{ty}) \cdot y\, dt.$$

For the second statement, let $f \in W^{1,1}_{\text{loc}}$, and set

$$T(\phi) = \int_{\mathbb{R}^d} \phi(x) f(x)\, dx.$$

Then $T \in \mathcal{D}'(\Omega)$, and $\partial_j T(\phi) = -T(\partial_j \phi) = -\int_{\mathbb{R}^d} \partial_j \phi f$. So

$$\partial_j T(\phi) = \int_{\mathbb{R}^d} \phi(x) \partial_j f(x)\, dx.$$

By the first statement,

$$\int_{\mathbb{R}^d} (\phi_y(x) - \phi(x)) f(x)\, dx = \int_0^1 \int_{\mathbb{R}^d} \sum_{j=1}^{d} y_j \phi_{ty}(x) \partial_j f(x)\, dx\, dt.$$

The left hand side is

$$\int_{\mathbb{R}^d} (\phi(x-y) - \phi(x)) f(x)\, dx,$$

and the left hand side is

$$\int_0^1 \int_{\mathbb{R}^d} \sum_{j=1}^{d} y_j \phi(x - ty) \partial_j f(x)\, dx\, dt.$$

If we make the change of variables $z = z - y$, then

$$\int_{\mathbb{R}^d} \phi(z)(f(z+y) - f(z))\, dz = \int_0^1 \int_{\mathbb{R}^d} \sum_{j=1}^{d} y_j \phi(z) \partial_j f(z + ty)\, dz\, dt.$$

Since ϕ is of compact support and $\partial_j f \in L^1_{\text{loc}}$ we check that we can apply Fubini's theorem to conclude that

$$\int_{\mathbb{R}^d} \phi(z)(f(z+y) - f(z))\, dz = \int_{\mathbb{R}^d} \phi(z) \left(\int_0^1 \nabla f(z+ty) \cdot y\, dt \right) dz.$$

By H older's inequality, this implies that $z \mapsto \int_0^1 \nabla f(z+ty) \cdot y\, dt \in L^1_{\text{loc}}(\mathbb{R}^d)$, and

$$f(z+y) - f(z) = \int_0^1 \nabla f(z+ty) \cdot y\, dt$$

for a.e. $z \in \mathbb{R}^d$. $\square$

Remark 28.1. Let $f \in C^1(\Omega)$, and set

$$T(\phi) = \int_\Omega f(x)\phi(x)\,dx), \qquad \phi \in C_c^\infty(\Omega).$$

Then $T \in \mathcal{D}'(\Omega)$, and

$$\partial_j T(\phi) = \int_\Omega \frac{\partial f}{\partial x_j}(x)\phi(x)\,dx,$$

where $\frac{\partial f}{\partial x_j}$ is the pointwise derivative.

This has a converse.

Theorem 28.2. *Let $g_1, \ldots, g_d \in C(\Omega)$, and let $T \in \mathcal{D}'(\Omega)$ be such that $\partial_j T = g_j$ for $j = 1, \ldots, d$. Then there exists $f \in C^1(\Omega)$ such that*

$$T(\phi) = \int_\Omega f(x)\phi(x)\,dx, \qquad \phi \in C_c^\infty(\Omega).$$

Then

$$g_j = \frac{\partial f}{\partial x_j}.$$

Corollary 28.1. *If Ω is connected, $T \in \mathcal{D}'(\Omega)$, and $\partial_j = 0$ for $j = 1, \ldots, d$, then there exists $C \in R$ such that*

$$T(\phi) = C \int_\Omega \phi(x)\,dx, \qquad \forall \phi \in C_c^\infty(\Omega).$$

29 Convolution of Distributions and Approximation of $W^{1,p}_{\text{loc}}$ Functions by C^∞ Functions

29.1 Convolution of distributions

If you solve $|Du| = 1$ with some boundary condition, it is unlikely that wou will find a solution in $C^1(\Omega)$. You will probably find a solution in $W^{1,1}_{\text{loc}}(\Omega)$. But we can approximate functions in $C^1(\Omega)$ by functions in $W^{1,1}_{\text{loc}}(\Omega)$. We can also approximate by functions in $C^\infty(\Omega)$. Oftentimes, we want to show that we have a solution in some bigger space and see if we can show it has extra properties that force it to be in a smaller, nicer space.

Let $\Omega \subseteq \mathbb{R}^d$ be an open set. If $\phi \in C_c^\infty(\Omega)$, we define $O_\phi = \{y \in \mathbb{R}^d : y + \text{supp}(\phi) \subseteq \Omega\}$. If $\psi \in L^1(O_\phi)$ is bounded, then

$$T(\psi * \phi) = \int_{O_\phi} \psi(y) T(\phi_y)\, dy.$$

for $T \in \mathcal{D}'(\Omega$ and $\phi_y(x) = \phi(x-y)$.

Given $j : A \subseteq \mathbb{R}^d \to \mathbb{R}$, we define $\tilde{h}L - A \to \mathbb{R}$ as $\tilde{j} = j(-x)$. If $T \in \mathcal{D}'(\mathbb{R}^d)$ and $j \in C_c^\infty(\mathbb{R}^d)$, we define $j * T : C_c^\infty(\mathbb{R}^d) \to \mathbb{R}$ as

$$j * T(\phi) = T(\tilde{j} * \phi).$$

Theorem 29.1. *Let $T \in \mathcal{D}'(\mathbb{R}^d)$, and let $j \in C_c^\infty(\mathbb{R}^d)$.*

1. *There exists $\psi \in C^\infty(\mathbb{R}^d)$ such that*

$$j * T(\phi) = \int_{\mathbb{R}^d} \phi(y)\psi(y)\, dy, \qquad \forall \phi \in C_c^\infty(\mathbb{R}^d)$$

and so $j * T \in \mathcal{D}'(\mathbb{R}^d)$.

2. *Further assume $\int_{\mathbb{R}^d} j(x)\, dx = 1$, and set $j_\varepsilon = \varepsilon^{-d} j(x/\varepsilon)$ for $x \in \mathbb{R}^d$. Then $(j_\varepsilon * T)_\varepsilon$ converges to T in $\mathcal{D}'(\mathbb{R}^d)$ as $\varepsilon \downarrow 0$.*

Remark 29.1. This shows that we have an embedding from $C^\infty(\Omega)$ into $\mathcal{D}'(\Omega)$ and that this class of functions is dense in $\mathcal{D}'(\Omega)$.

Proof. Note that $O_{\tilde{j}} = \{y \in \mathbb{R}^d : y + \text{supp}(\tilde{j}) \in \mathbb{R}^d\} = \mathbb{R}^d$. By the formula for distributions applied to convolutions, we get

$$j * T(\phi) = T(\tilde{j} * \phi) = \int_{O_{\tilde{j}}} \psi(y) T(\tilde{j}_y)\, dy.$$

Since $\tilde{j} \in C_c^\infty(\mathbb{R}^d)$, $y \mapsto T(\tilde{j}y)$ is of class C^∞.

For the second statement, if $\phi \in C_c^\infty(\mathbb{R}^d)$,

$$\lim_{\varepsilon \to 0} j_\varepsilon * T(\phi) = \lim_{\varepsilon \to 0} T(\tilde{j}_\varepsilon * \phi) = T(\phi)$$

since $\tilde{j}_\varepsilon * \phi$ converges to ϕ in $C_c^\infty(\mathbb{R}^d)$. $\square$

29.2 Approximation of $W^{1,p}_{\mathrm{loc}}$ functions by C^∞ functions

Theorem 29.2. *Let $1 \le p < \infty$, and let $f \in W^{1,p}_{\mathrm{loc}}(\Omega)$. Then for every open, bounded $O \subseteq \mathbb{R}^d$ such that $\overline{O} \subseteq \Omega$, there exists $(f^k)_k \subseteq C^\infty(O)$ such that*

$$\lim_{k\to\infty} \|f - f^k\|_{W^{1,p}(O)} = 0.$$

Remark 29.2. This is equivalent to saying that $f_k \in C_0^\infty(O) \cap W^{1,p}_{\mathrm{loc}}(O)$.

Proof. Let $\delta = \mathrm{dist}(\overline{O}, \Omega^c) > 0$. Let $j = C_c^\infty(\mathbb{R}^d)$ be such that $\int_{\mathbb{R}^d} j(x)\,dx = 1$ and $\mathrm{supp}(j) = \overline{B_1(0)}$. Set $j_\varepsilon(x) = \varepsilon^{-d} j(x/\varepsilon)$ for $0 < \varepsilon < \delta/3$. Note that $j_\varepsilon * f$, $j_\varepsilon * \nabla f$ are well-defined on O for these ε. We have $j_\varepsilon * f \in C^\infty(O)$ and that

$$0 = \lim_{\varepsilon\to 0} \|j_\varepsilon * f - f\|_{L^p(O)} = \lim_{\varepsilon\to 0} \|j_\varepsilon * \nabla f - \nabla f\|_{L^p(O)}.$$

Set $f^k = j_{1/k} * f$ to conclude the proof. $\square$